Untitled/unsorted collection of math notes

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What is this?

This a collection of notes gathered here with an aim to offload my other books ("Reverse Engineering for Beginners"[4]and "SAT/SMT by Example"[5]) from mathematical theory, which still can be of interest for a general audience of programmers.

It can be said, this is a *companion book*.

Also, random posts from my blog.

So, nothing special.

Started in August-2015.

Thanks

Henry Legge.

[4]`https://beginners.re/`
[5]`https://sat-smt.codes/`

Chapter 1

Unsorted parts

1.1 Fencepost error / off-by-one error

Here's yet another illustration of such errors.

You are riding by metro/tram/bus and your friend asking you, "how many stations/stops shall we pass before your the destination station?"

For people striving for accuracy and exact sciences in general, such questions are like scraping a chalkboard. It is even worse if, at the moment of the question, the train/tram/bus is at a stop.

To answer such a question, you need to clarify — shall we count stops at stations or passings between them? Do we take into account the current stop and/or the final one?

Without answers to these questions, such errors appear.

Also, an example from Kurt Vonnegut's Cat's Cradle, the very beginning of the book:

> When I was a younger man–two wives ago, 250,000 cigarettes ago, 3,000 quarts of booze ago.

Go figure, how many ways there are exist to determine that point in time ("two wives ago"). Considering that Kurt Vonnegut at the moment of writing was 1) married; 2) not married.

Also, often, a phrase in advertisement like "... [something] is valid from January 10th to 20th". Without the addition of the word "inclusive", it is not clear if January 20th is included in this interval or not? Is this a half-closed interval or a closed interval?

See also: http://www.catb.org/jargon/html/O/off-by-one-error.html, http://www.catb.org/jargon/html/F/fencepost-error.html.

Now an excerpt from the amusing article by Denys Wilkinson:

> Communications direct to the projectionist are always good and should be made in such a manner that it is not immediately clear whether the Slidesman is addressing the projectionist or the audience. Absurd complication in the instructions must be avoided. The Slidesman uses something like: 'After the next slide but two I shall want to look again at the last one but four.' After the next slide: 'I meant of course that slide which was then going to be the last one but four after I had indeed had that which was going to be the next but two, not that which was then the last but four.' Follow this by skipping one slide.

[His article can be found in *A Random Walk In Science — Eric Mendoza*[1].]

[1]https://archive.org/details/EricMendozaEd.ARandomWalkInScience_201406/

1.2 GCD and LCM

1.2.1 GCD

What is Greatest common divisor (GCD[2])?

Let's suppose, you want to cut a rectangle by squares. What is maximal square could be?

For a 14*8 rectangle, this is 2*2 square:

```
**************
**************
**************
**************
**************
**************
**************
**************
```

->

```
** ** ** ** ** ** **
** ** ** ** ** ** **

** ** ** ** ** ** **
** ** ** ** ** ** **

** ** ** ** ** ** **
** ** ** ** ** ** **

** ** ** ** ** ** **
** ** ** ** ** ** **
```

What for 14*7 rectangle? It's 7*7 square:

```
**************
**************
**************
**************
**************
**************
**************
```

->

```
******* *******
******* *******
******* *******
******* *******
******* *******
******* *******
******* *******
```

14*9 rectangle? 1, i.e., smallest possible.

```
**************
**************
**************
**************
**************
**************
**************
```

[2]Greatest Common Divisor

```
**************
**************
**************
```

GCD of coprimes is 1.

GCD is also a common set of factors of several numbers. This we can see in Mathematica:

```
In[]:= FactorInteger[300]
Out[]= {{2, 2}, {3, 1}, {5, 2}}

In[]:= FactorInteger[333]
Out[]= {{3, 2}, {37, 1}}

In[]:= GCD[300, 333]
Out[]= 3
```

I.e., $300 = 2^2 \cdot 3 \cdot 5^2$ and $333 = 3^2 \cdot 37$ and $GCD = 3$, which is smallest factor.

Or:

```
In[]:= FactorInteger[11*13*17]
Out[]= {{11, 1}, {13, 1}, {17, 1}}

In[]:= FactorInteger[7*11*13*17]
Out[]= {{7, 1}, {11, 1}, {13, 1}, {17, 1}}

In[]:= GCD[11*13*17, 7*11*13*17]
Out[]= 2431

In[]:= 11*13*17
Out[]= 2431
```

(Common factors are 11, 13 and 17, so $GCD = 11 \cdot 13 \cdot 17 = 2431$.)

Listing 1.1: Another example from the *Structure and Interpretation of Computer Programs*

```
The greatest common divisor (GCD) of two integers a and b is defined to be the
   largest integer that divides both a and b with no remainder.
```

Listing 1.2: And one more

```
To reduce a rational number to lowest terms, we must divide both the numerator and
   the denominator by their GCD. For example, 16/28 reduces to 4/7.
```

Another example:

```
Debate among kernel developers eventually resulted in the software clock rate
   becoming a configurable kernel option (under Processor type and features, Timer
   frequency). Since kernel 2.6.13, the clock rate can be set to 100, 250 (the
   default), or 1000 hertz, giving jiffy values of 10, 4, and 1 milliseconds,
   respectively. Since kernel 2.6.20, a further frequency is available: 300 hertz, a
   number that divides evenly for two common video frame rates: 25 frames per second
   (PAL) and 30 frames per second (NTSC).
```

(Michael Kerrisk – The Linux Programming Interface)

Those of us who remember old-school modem days of 1990s may remember COM port speeds in baud: https://en.wikipedia.org/wiki/Serial_port#Speed, https://en.wikipedia.org/wiki/Modem#Evolution_of_dial-up_speeds

```
In[]:= GCD[300, 1200, 2400, 4800, 7200, 9600, 14400, 19200, 28800, 33600, 38400,
    56700, 115200]
Out[]:= 300
```

1.2.2 Oculus VR Flicks and GCD

I've found this:

```
A flick (frame-tick) is a very small unit of time. It is 1/705600000 of a second,
    exactly.

1 flick = 1/705600000 second

This unit of time is the smallest time unit which is LARGER than a nanosecond,
and can in integer quantities exactly represent a single frame duration for
24hz, 25hz, 30hz, 48hz, 50hz, 60hz, 90hz, 100hz, 120hz, and also 1/1000 divisions of
    each.
This makes it suitable for use via std::chrono::duration and std::ratio
for doing timing work against the system high resolution clock, which is in
    nanoseconds,
but doesn't get slightly out of sync when doing common frame rates.

In order to accommodate media playback, we also support some common audio sample
    rates as well.
This list is not exhaustive, but covers the majority of digital audio formats.
They are 8kHz, 16kHz, 22.05kHz, 24kHz, 32kHz, 44.1kHz, 48kHz, 88.2kHz, 96kHz, and 192
    kHz.
While humans can't hear higher than 48kHz, the higher sample rates are used
for working audio files which might later be resampled or retimed.

The NTSC variations (~29.97, etc) are actually defined as 24 * 1000/1001 and
30 * 1000/1001, which are impossible to represent exactly in a way where 1 second is
    exact,
so we don't bother - they'll be inexact in any circumstance.

Details

    1/24 fps frame: 29400000 flicks
    1/25 fps frame: 28224000 flicks
    1/30 fps frame: 23520000 flicks
    1/48 fps frame: 14700000 flicks
    1/50 fps frame: 14112000 flicks
    1/60 fps frame: 11760000 flicks
    1/90 fps frame: 7840000 flicks
    1/100 fps frame: 7056000 flicks
    1/120 fps frame: 5880000 flicks
    1/8000 fps frame: 88200 flicks
    1/16000 fps frame: 44100 flicks
    1/22050 fps frame: 32000 flicks
    1/24000 fps frame: 29400 flicks
    1/32000 fps frame: 22050 flicks
    1/44100 fps frame: 16000 flicks
    1/48000 fps frame: 14700 flicks
    1/88200 fps frame: 8000 flicks
    1/96000 fps frame: 7350 flicks
    1/192000 fps frame: 3675 flicks
```

(https://github.com/OculusVR/Flicks)

Where the number came from? Let's enumerate all possible time intervals they want to use and find GCD using Mathematica:

```
In[]:= GCD[1/24, 1/24000, 1/25, 1/25000, 1/30, 1/30000, 1/48, 1/50,
 1/50000, 1/60, 1/60000, 1/90, 1/90000, 1/100, 1/100000, 1/120,
 1/120000, 1/8000, 1/16000, 1/22050, 1/24000, 1/32000, 1/44100,
 1/48000, 1/88200, 1/96000, 1/192000]

Out[]= 1/705600000
```

Rationale: you may want to play a video with $\frac{1}{50}$ fps and, simultaneously, play audio with 96kHz sampling rate. Given that, you can change video frame after each 14112000 flicks and change one audio sample after each 7350 flicks. Use any other video fps and any audio sampling rate and you will have all time periods as integer numbers. No ratios any more.

On contrary, one nanosecond wouldn't fit: try to represent 1/30 second in nanoseconds, this is (still) ratio: 33333.33333... nanoseconds.

1.2.3 LCM

Many people use LCM[3] in school. Sum up $\frac{1}{4}$ and $\frac{1}{6}$. To find an answer mentally, you ought to find Lowest Common Denominator, which can be 4*6=24. Now you can sum up $\frac{6}{24} + \frac{4}{24} = \frac{10}{24}$.

But the lowest denominator is also a LCM. LCM of 4 and 6 is 12: $\frac{3}{12} + \frac{2}{12} = \frac{5}{12}$.

This happens — you stuck in traffic jams, there are two vehicles in front of you, and you see how yellow turn signals blinking, out of sync. You wonder, how many blinks you have to wait so that they will blink at once?

If the first signal blinking with period of 0.6s (or 600ms) and the second is 0.5s (or 500ms), this problem can be solved graphically:

```
*****|*****|*****|*****|*****| 600ms
****|****|****|****|****|****| 500ms
```

(One character for 100ms.)

They will blink synchronously once in each 3 seconds (or 3000ms) period. This is exactly what least common multiple is.

You can extend this problem to 3 turn signals.

File copying routine

Buffer: A storage device used to compensate for a difference in data rate of data flow or time of occurrence of events, when transmitting data from one device to another.

Clarence T. Jones, S. Percy Jones – Patrick-Turner's Industrial Automation Dictionary

In GNU coreutils, we can find that LCM is used to find optimal buffer size, if buffer sizes in input and output files are differ. For example, input file has buffer of 4096 bytes, and output is 6144. Well, these sizes are somewhat suspicious. I made up this example. Nevertheless, LCM of 4096 and 6144 is 12288. This is a buffer size you can allocate, so that you will minimize number of read/write operations during copying.

https://github.com/coreutils/coreutils/blob/4cb3f4faa435820dc99c36b30ce93c7d01501f65/src/copy.c#L1246.
https://github.com/coreutils/coreutils/blob/master/gl/lib/buffer-lcm.c.

[3]Least Common Multiple

1.3 Signed numbers: two's complement

There are several methods for representing signed numbers, but "two's complement" is the most popular one in computers.

Here is a table for some byte values:

binary	hexadecimal	unsigned	signed
01111111	0x7f	127	127
01111110	0x7e	126	126
...			
00000110	0x6	6	6
00000101	0x5	5	5
00000100	0x4	4	4
00000011	0x3	3	3
00000010	0x2	2	2
00000001	0x1	1	1
00000000	0x0	0	0
11111111	0xff	255	-1
11111110	0xfe	254	-2
11111101	0xfd	253	-3
11111100	0xfc	252	-4
11111011	0xfb	251	-5
11111010	0xfa	250	-6
...			
10000010	0x82	130	-126
10000001	0x81	129	-127
10000000	0x80	128	-128

The difference between signed and unsigned numbers is that if we represent `0xFFFFFFFE` and `0x00000002` as unsigned, then the first number (4294967294) is bigger than the second one (2). If we represent them both as signed, the first one becomes -2, and it is smaller than the second (2).

That is the reason why conditional jumps in x86 are present both for signed (e.g. `JG`, `JL`) and unsigned (`JA`, `JB`) operations.

For the sake of simplicity, this is what one needs to know:

- Numbers can be signed or unsigned.
- C/C++ signed types:
  - `int64_t` (-9,223,372,036,854,775,808 .. 9,223,372,036,854,775,807) (- 9.2.. 9.2 quintillions) or `0x8000000000000000..0x7FFFFFFFFFFFFFFF`),
  - `int` (-2,147,483,648..2,147,483,647 (- 2.15.. 2.15Gb) or `0x80000000..0x7FFFFFFF`),
  - `char` (-128..127 or `0x80..0x7F`),
  - `ssize_t`.

  Unsigned:

  - `uint64_t` (0..18,446,744,073,709,551,615 (18 quintillions) or `0..0xFFFFFFFFFFFFFFFF`),
  - `unsigned int` (0..4,294,967,295 (4.3Gb) or `0..0xFFFFFFFF`),
  - `unsigned char` (0..255 or `0..0xFF`),
  - `size_t`.
- Signed types have the sign in the MSB[4]: 1 means "minus", 0 means "plus".
- Promoting to a larger data types is simple: get the leftmost bit and fill all other newly created bits with it.

[4]Most Significant Bit

- Negation is simple: just invert all bits and add 1.

  We can keep in mind that a number of inverse sign is located on the opposite side at the same proximity from zero. The addition of one is needed because zero is present in the middle.

- The addition and subtraction operations work well for both signed and unsigned values. But for multiplication and division operations, x86 has different instructions: `IDIV`/`IMUL` for signed and `DIV`/`MUL` for unsigned.

- Here are some more x86 instructions that work with signed numbers:
  `CBW/CWD/CWDE/CDQ/CDQE`, `MOVSX`, `SAR`.

A table of some negative and positive values (1.3) looks like thermometer with Celsius scale. This is why addition and subtraction works equally well for both signed and unsigned numbers: if the first addend is represented as mark on thermometer, and one need to add a second addend, and it's positive, we just shift mark up on thermometer by the value of second addend. If the second addend is negative, then we shift mark down to absolute value of the second addend.

Addition of two negative numbers works as follows. For example, we need to add -2 and -3 using 16-bit registers. -2 and -3 is 0xfffe and 0xfffd respectively. If we add these numbers as unsigned, we will get 0xfffe+0xfffd=0x1fffb. But we work on 16-bit registers, so the result is *cut off*, the first 1 is dropped, 0xfffb is left, and this is -5. This works because -2 (or 0xfffe) can be represented using plain English like this: "2 lacks in this value up to maximal value in 16-bit register + 1". -3 can be represented as "...3 lacks in this value up to ...". Maximal value of 16-bit register + 1 is 0x10000. During addition of two numbers and *cutting off* by 2^{16} modulo, $2 + 3 = 5$ *will be lacking*.

1.3.1 Couple of additions about two's complement form

Exercise 2-1. Write a program to determine the ranges of `char`, `short`, `int`, and `long` variables, both `signed` and `unsigned`, by printing appropriate values from standard headers and by direct computation.

Brian W. Kernighan, Dennis M. Ritchie, *The C Programming Language*, 2ed, (1988)

Getting maximum number of some *word*

Maximum unsigned number is just a number where all bits are set: *0xFF....FF* (this is -1 if the *word* is treated as signed integer). So you take a *word*, set all bits and get the value:

```
#include <stdio.h>

int main()
{
        unsigned int val=~0; // change to "unsigned char" to get maximal value for the
            unsigned 8-bit byte
        // 0-1 will also work, or just -1
        printf ("%u\n", val); // %u for unsigned
};
```

This is 4294967295 for 32-bit integer.

Getting minimum number for some signed *word*

Minimum signed number is encoded as *0x80....00*, i.e., most significant bit is set, while others are cleared. Maximum signed number is encoded in the same way, but all bits are inverted: *0x7F....FF*.

Let's shift a lone bit left until it disappears:

```
#include <stdio.h>

int main()
{
        signed int val=1; // change to "signed char" to find values for signed byte
        while (val!=0)
        {
                printf ("%d %d\n", val, ~val);
                val=val<<1;
        };
};
```

Output is:

```
...

536870912 -536870913
1073741824 -1073741825
-2147483648 2147483647
```

Two last numbers are minimum and maximum signed 32-bit *int* respectively.

1.3.2 -1

Now you see that -1 is when all bits are set. Often, you can find the -1 constant in all sorts of code, where a constant with all bits set are needed, for example, a mask.

1.4 Mandelbrot set

You know, if you magnify the coastline, it still looks like a coastline, and a lot of other things have this property. Nature has recursive algorithms that it uses to generate clouds and Swiss cheese and things like that.

Donald Knuth, interview (1993)

Mandelbrot set is a fractal, which exhibits self-similarity.

When you increase scale, you see that this characteristic pattern repeating infinitely.

1.4.1 A word about complex numbers

A complex number is a number that consists of two parts—real (Re) and imaginary (Im).

The complex plane is a two-dimensional plane where any complex number can be placed: the real part is one coordinate and the imaginary part is the other.

Some basic rules we have to keep in mind:

- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

  In other words:

  $\mathrm{Re}(sum) = \mathrm{Re}(a) + \mathrm{Re}(b)$

  $\mathrm{Im}(sum) = \mathrm{Im}(a) + \mathrm{Im}(b)$

- Multiplication: $(a + bi)(c + di) = (ac - bd) + (bc + ad)i$

  In other words:

  $\mathrm{Re}(product) = \mathrm{Re}(a) \cdot \mathrm{Re}(c) - \mathrm{Re}(b) \cdot \mathrm{Re}(d)$

  $\mathrm{Im}(product) = \mathrm{Im}(b) \cdot \mathrm{Im}(c) + \mathrm{Im}(a) \cdot \mathrm{Im}(d)$

- Square: $(a + bi)^2 = (a + bi)(a + bi) = (a^2 - b^2) + (2ab)i$

  In other words:

  $\mathrm{Re}(square) = \mathrm{Re}(a)^2 - \mathrm{Im}(a)^2$

$\text{Im}(square) = 2 \cdot \text{Re}(a) \cdot \text{Im}(a)$

1.4.2 How to draw the Mandelbrot set

The Mandelbrot set is a set of points for which the $z_{n+1} = {z_n}^2 + c$ recursive sequence (where z and c are complex numbers and c is the starting value) does not approach infinity.

In plain English language:

- Enumerate all points on screen.
- Check if the specific point is in the Mandelbrot set.
- Here is how to check it:
  - Represent the point as a complex number.
  - Calculate the square of it.
  - Add the starting value of the point to it.
  - Does it go off limits? If yes, break.
  - Move the point to the new place at the coordinates we just calculated.
  - Repeat all this for some reasonable number of iterations.
- The point is still in limits? Then draw the point.
- The point has eventually gone off limits?
  - (For a black-white image) do not draw anything.
  - (For a colored image) transform the number of iterations to some color. So the color shows the speed with which point has gone off limits.

Here is Pythonesque algorithm for both complex and integer number representations:

Listing 1.3: For complex numbers

```
def check_if_is_in_set(P):
    P_start=P
    iterations=0

    while True:
        if (P>bounds):
            break
        P=P^2+P_start
        if iterations > max_iterations:
            break
        iterations++

    return iterations

# black-white
for each point on screen P:
    if check_if_is_in_set (P) < max_iterations:
        draw point

# colored
for each point on screen P:
    iterations = if check_if_is_in_set (P)
    map iterations to color
    draw color point
```

The integer version is where the operations on complex numbers are replaced with integer operations according to the rules which were explained above.

Listing 1.4: For integer numbers

```
def check_if_is_in_set(X, Y):
    X_start=X
    Y_start=Y
    iterations=0

    while True:
        if (X^2 + Y^2 > bounds):
            break
        new_X=X^2 - Y^2 + X_start
        new_Y=2*X*Y + Y_start
        if iterations > max_iterations:
            break
        iterations++

    return iterations

# black-white
for X = min_X to max_X:
    for Y = min_Y to max_Y:
        if check_if_is_in_set (X,Y) < max_iterations:
            draw point at X, Y

# colored
for X = min_X to max_X:
    for Y = min_Y to max_Y:
        iterations = if check_if_is_in_set (X,Y)
        map iterations to color
        draw color point at X,Y
```

Here is also a C# source which is present in the Wikipedia article [5], but we'll modify it so it will print the iteration numbers instead of some symbol [6]:

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;

namespace Mnoj
{
    class Program
    {
        static void Main(string[] args)
        {
            double realCoord, imagCoord;
            double realTemp, imagTemp, realTemp2, arg;
            int iterations;
            for (imagCoord = 1.2; imagCoord >= -1.2; imagCoord -= 0.05)
            {
                for (realCoord = -0.6; realCoord <= 1.77; realCoord += 0.03)
                {
                    iterations = 0;
                    realTemp = realCoord;
                    imagTemp = imagCoord;
                    arg = (realCoord * realCoord) + (imagCoord * imagCoord);
                    while ((arg < 2*2) && (iterations < 40))
                    {
                        realTemp2 = (realTemp * realTemp) - (imagTemp * imagTemp) -
                            realCoord;
```

[5] wikipedia

[6] Here is also the executable file: https://αβγ.ελ/current_tree/unsorted/mandel/dump_iterations.exe

```
                        imagTemp = (2 * realTemp * imagTemp) - imagCoord;
                        realTemp = realTemp2;
                        arg = (realTemp * realTemp) + (imagTemp * imagTemp);
                        iterations += 1;
                    }
                    Console.Write("{0,2:D} ", iterations);
                }
                Console.Write("\n");
            }
            Console.ReadKey();
        }
    }
}
```

Here is the resulting file, which is too wide to be included here:
https://αβγ.ελ/current_tree/unsorted/mandel/result.txt.

The maximal number of iterations is 40, so when you see 40 in this dump, it means that this point has been wandering for 40 iterations but never got off limits.

A number n less than 40 means that point remained inside the bounds only for n iterations, then it went outside them.

There is a cool demo available at http://demonstrations.wolfram.com/MandelbrotSetDoodle/, which shows visually how the point moves on the plane at each iteration for some specific point. Here are two screenshots.

First, we've clicked inside the yellow area and saw that the trajectory (green line) eventually swirls at some point inside:

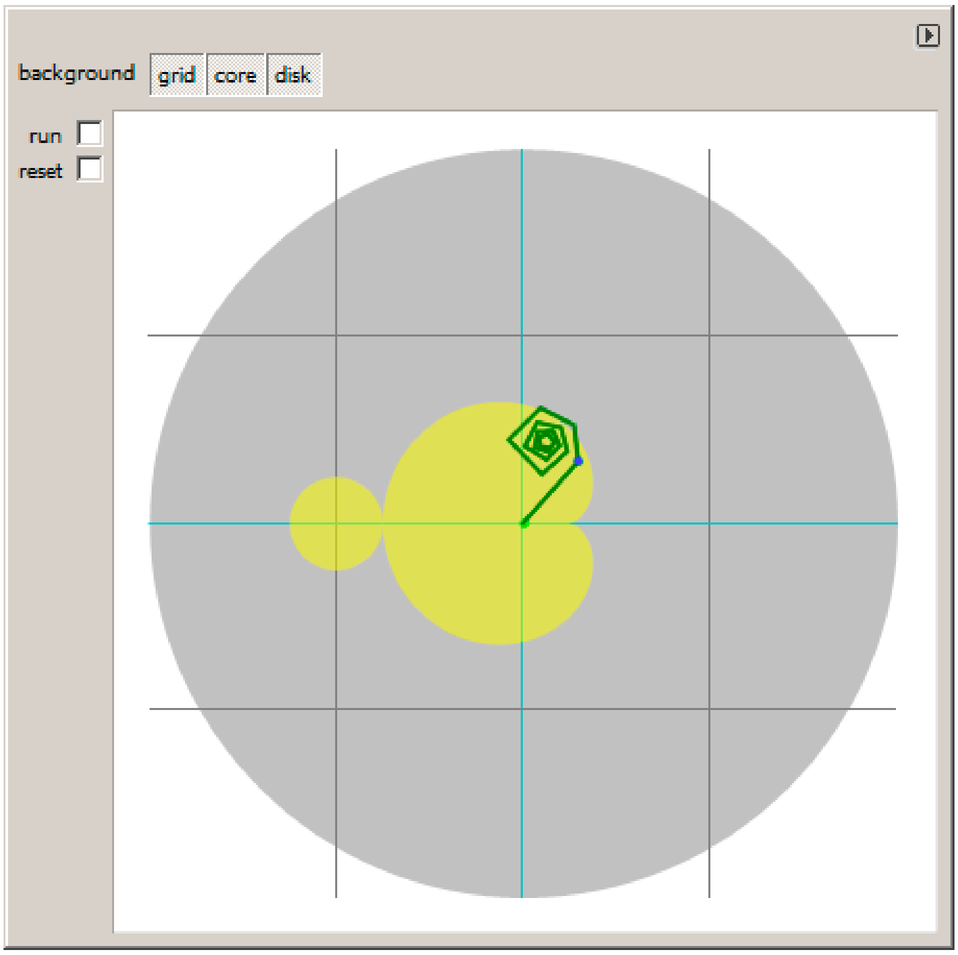


Figure 1.1: Click inside yellow area

This implies that the point we've clicked belongs to the Mandelbrot set.

Then we've clicked outside the yellow area and saw a much more chaotic point movement, which quickly went off bounds:

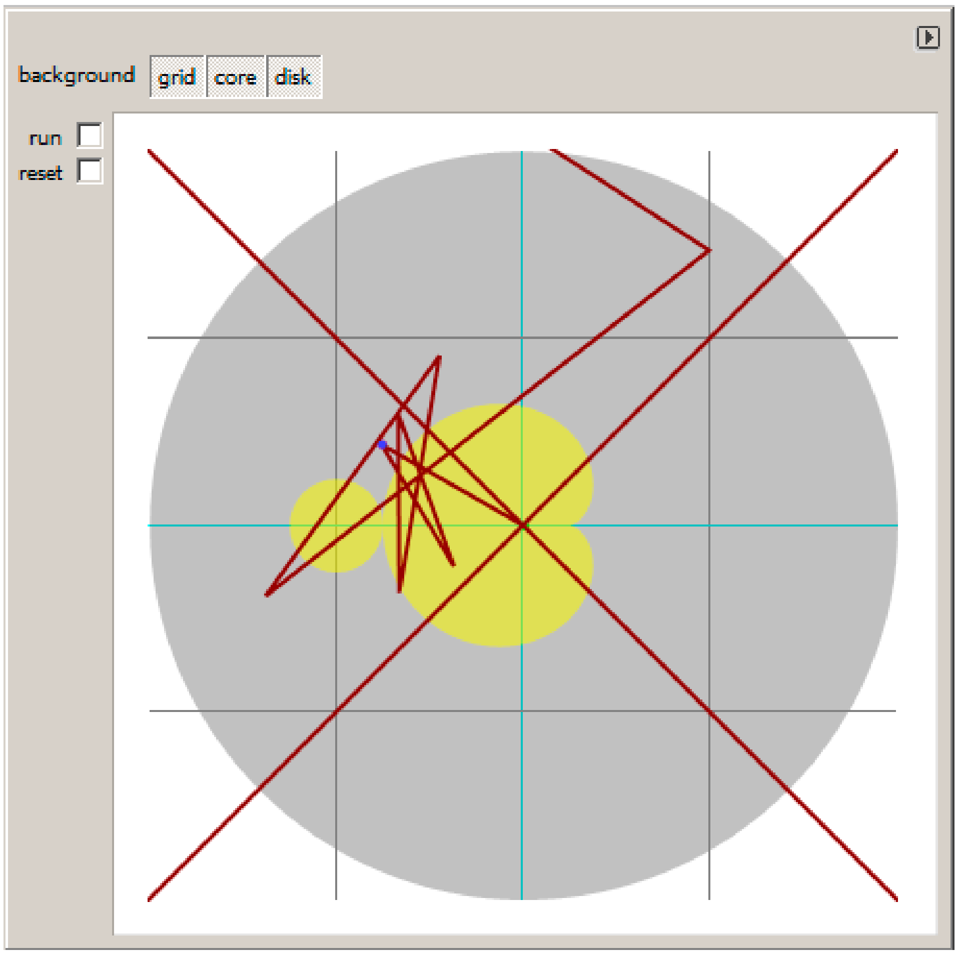


Figure 1.2: Click outside yellow area

This means the point doesn't belong to Mandelbrot set.

Another good demo is available here:
http://demonstrations.wolfram.com/IteratesForTheMandelbrotSet/.

1.5 Pythagorean theorem

There is a popular example, how to determine π value using Monte Carlo method.

Just google: "monte carlo pythagorean". For example: 1, 2.

This is my example. In fact, it determines area of the circle for a given $2 * 2 = 4$ square. This equals to π.

Listing 1.5: The source code.

```
#!/usr/bin/env python3
import random, math

inside = 0
total = 10000000

for i in range(0, total):
    x = random.random()
    y = random.random()
    if math.sqrt(x**2 + y**2) < 1.0:
    # will also work:
    # if x**2 + y**2 < 1.0:
        inside += 1

area = float(inside) / total*(2**2)
print("area of square", 2**2)
print("area of circle", area)
```

Listing 1.6: The result. Close to π.

```
area of square 4
area of circle 3.1414132
```

You can remove $\sqrt{x}$ operation, because all you need to know if $\sqrt{x} < 1$, and this is the same as $x < 1$. (If the first condition holds, the second holds as well and vice versa.)

I couldn't stand the itch and I tried higher dimensions.

It works, because Pythagorean theorem generalizes to higher dimensions [7]: $x^2 + y^2 + z^2 = r^2$, where x/y/z are 3D coordinates and r is radius. This is true for 3D sphere.

Listing 1.7: The source code.

```
#!/usr/bin/env python3
import random, math

inside = 0
total = 10000000

for i in range(0, total):
    x = random.random()
    y = random.random()
    z = random.random()
    if math.sqrt(x**2 + y**2 + z**2) < 1.0:
    # will also work:
    # if x**2 + y**2 + z**2 < 1.0:
        inside += 1

area = float(inside) / total*(2**3)
print("volume of cube", 2**3)
print("volume sphere", area)
```

Listing 1.8: The result.

```
volume of cube 8
volume sphere 4.1884288
```

With some effort, it's possible to tabulate this function and to deduce the volume formula for sphere: $\frac{4\pi}{3}r^3$.

Now extend this to tesseract/4-sphere and 5-cube/5-sphere:

Listing 1.9: The source code.

```
#!/usr/bin/env python3
import random, math

inside = 0
total = 10000000

for i in range(0, total):
    x = random.random()
    y = random.random()
    z = random.random()
    t = random.random()
    if math.sqrt(x**2 + y**2 + z**2 + t**2) < 1.0:
```

[7]For example: https://www.mathsisfun.com/geometry/pythagoras-3d.html

```
    # will also work:
    # if x**2 + y**2 + z**2 + t**2 < 1.0:
        inside += 1

area = float(inside) / total*(2**4)
print("volume of tesseract", 2**4)
print("volume 4-sphere", area)
```

Listing 1.10: The result.

```
volume of tesseract 16
volume 4-sphere 4.9368784
```

Listing 1.11: The source code.

```
#!/usr/bin/env python3
import random, math

inside = 0
total = 10000000

for i in range(0, total):
    x = random.random()
    y = random.random()
    z = random.random()
    t = random.random()
    v = random.random()
    if math.sqrt(x**2 + y**2 + z**2 + t**2 + v**2) < 1.0:
    # will also work:
    # if x**2 + y**2 + z**2 + t**2 + v**2 < 1.0:
        inside += 1

area = float(inside) / total*(2**5)
print("volume of 5-cube", 2**5)
print("volume 5-sphere", area)
```

Listing 1.12: The result.

```
volume of 5-cube 32
volume 5-sphere 5.2671776
```

Again, it's possible to deduce formulas for 4-sphere and 5-sphere, which are listed here: https://en.wikipedia.org/wiki/Volume_of_an_n-ball.

Isn't it cool? We can imagine 4-sphere or 5-sphere (well, most of us), but we can calculate volume of these spheres and deduce a formula for them.

As checked in Wolfram Mathematica, volumes for circle, sphere, 4-sphere, 5-sphere for $r = 1$ (unit circle):

Listing 1.13: Wolfram Mathematica

```
In[8]:= r = 1;

In[10]:= Pi*r^2
Out[10]= \[Pi]

In[12]:= (4/3)*Pi*r^3 // N
Out[12]= 4.18879

In[13]:= ((Pi^2)/2)*r^4 // N
Out[13]= 4.9348

In[14]:= ((8*Pi^2)/15)*r^5 // N
Out[14]= 5.26379
```

Chapter 2

Boolean algebra

2.1 XOR (exclusive OR)

XOR is widely used when one needs just to flip specific bit(s). Indeed, the XOR operation applied with 1 effectively inverts a bit:

input A	input B	output
0	0	0
0	1	1
1	0	1
1	1	0

And vice-versa, the XOR operation applied with 0 does nothing, i.e., it's an idle operation. This is a very important property of the XOR operation.

2.1.1 Logical difference

In Cray-1 supercomputer (1976-1977) manual [1], you can find XOR instruction was called *logical difference*.

Indeed, XOR(a,b)=1 if a!=b.

2.1.2 Everyday speech

XOR operation present in common everyday speech. When someone asks "please buy apples or bananas", this usually means "buy the first object or the second, but not both"—this is exactly exclusive OR, because logical OR would mean "both objects are also fine".

Some people suggest "and/or" should be used in everyday speech to make emphasis that logical OR is used instead of exclusive OR: https://en.wikipedia.org/wiki/And/or.

2.1.3 Encryption

XOR is heavily used in both amateur[2] and *real* encryption (at least in *Feistel network*).

XOR is very useful here because: $cipher\_text = plain\_text \oplus key$ and then: $(plain\_text \oplus key) \oplus key = plain\_text$.

2.1.4 RAID4

RAID4 offers a very simple method to protect hard disks. For example, there are several disks (D_1, D_2, D_3, etc.) and one parity disk (P). Each bit/byte written to parity disk is calculated and written on-fly:

$$P = D_1 \oplus D_2 \oplus D_3 \tag{2.1}$$

If any of disks is failed, for example, D_2, it's restored using the very same way:

[1] http://www.bitsavers.org/pdf/cray/CRAY-1/HR-0004-CRAY_1_Hardware_Reference_Manual-PRELIMINARY-1975.OCR.pdf

[2] "SAT/SMT by Example"[3] has some examples.

$$D_2 = D_1 \oplus P \oplus D_3 \tag{2.2}$$

If parity disk failed, it is restored using 2.1 way. If two of any disks are failed, then it wouldn't be possible to restore both.

RAID5 is more advanced, but this XOR property is still exploited there.

That's why RAID controllers has hardware "XOR accelerators" helping to XOR large chunks of written data on-fly. When computers get faster and faster, it now can be done at software level, using SIMD[4].

2.1.5 XOR swap algorithm

Hard to believe, but this code swaps the values in EAX and EBX without aid of any other additional register or memory cell:

```
xor eax, ebx
xor ebx, eax
xor eax, ebx
```

Let's find out, how it works. First, we will rewrite it to step aside from x86 assembly language:

```
X = X XOR Y
Y = Y XOR X
X = X XOR Y
```

What X and Y has at each step? Just keep in mind the simple rule: $(X \oplus Y) \oplus Y = X$ for any values of X and Y.

Let's see, X after 1st step has $X \oplus Y$; Y after 2nd step has $Y \oplus (X \oplus Y) = X$; X after 3rd step has $(X \oplus Y) \oplus X = Y$.

Hard to say if anyone should use this trick, but it servers as a good demonstration example of XOR properties.

Wikipedia article[5] has also yet another explanation: addition and subtraction operations can be used instead of XOR:

```
X = X + Y
Y = X - Y
X = X - Y
```

Let's see: X after 1st step has $X + Y$; Y after 2nd step has $X + Y - Y = X$; X after 3rd step has $X + Y - X = Y$.

2.1.6 XOR linked list

Doubly linked list is a list in which each element has link to the previous element and to the next one. Hence, it's very easy to traverse list backwards or forward.

So each element has two pointers. Is it possible, perhaps in environment of small RAM[6] footprint, to preserve all functionality with one pointer instead of two? Yes, if it a value of $prev \oplus next$ will be stored in this memory cell, which is usually called "link".

Maybe, we could say that address to the previous element is "encrypted" using address of next element and otherwise: next element address is "encrypted" using previous element address.

When we traverse this list forward, we always know address of the previous element, so we can "decrypt" this field and get address of the next element. Likewise, it's possible to traverse this list backwards, "decrypting" this field using next element's address.

But it's not possible to find address of previous or next element of some specific element without knowing address of the first one.

Couple of things to complete this solution: first element will have address of next element without any XOR-ing, last element will have address of previous element without any XOR-ing.

Now let's sum it up. This is example of doubly linked list of 5 elements. A_x is address of element.

[4] Single Instruction, Multiple Data
[5] https://en.wikipedia.org/wiki/XOR_swap_algorithm
[6] Random-Access Memory

address	*link* field contents
A_0	A_1
A_1	$A_0 \oplus A_2$
A_2	$A_1 \oplus A_3$
A_3	$A_2 \oplus A_4$
A_4	A_3

And again, hard to say if anyone should use this tricky hacks, but this is also a good demonstration of XOR properties. As with XOR swap algorithm, Wikipedia article about it also offers way to use addition or subtraction instead of XOR: https://en.wikipedia.org/wiki/XOR_linked_list.

2.1.7 Switching value trick

... found in Jorg Arndt — Matters Computational / Ideas, Algorithms, Source Code [7].

You want a variable to be switching between 123 and 456. You may write something like:

```
if (a==123)
    a=456;
else
    a=123;
```

But this can be done using a single operation:

```
#include <stdio.h>

int main()
{
        int a=123;
#define C 123^456

        a=a^C;
        printf ("%d\n", a);
        a=a^C;
        printf ("%d\n", a);
        a=a^C;
        printf ("%d\n", a);
};
```

It works because $123 \oplus 123 \oplus 456 = 0 \oplus 456 = 456$ and $456 \oplus 123 \oplus 456 = 456 \oplus 456 \oplus 123 = 0 \oplus 123 = 123$.

One can argue, worth it using or not, especially keeping in mind code readability. But this is yet another demonstration of XOR properties.

2.1.8 Zobrist hashing / tabulation hashing

If you work on a chess engine, you traverse a game tree many times per second, and often, you can encounter the same position, which has already been processed.

So you have to use a method to store already calculated positions somewhere. But chess position can require a lot of memory, and a hash function would be used instead.

Here is a way to compress a chess position into 64-bit value, called Zobrist hashing:

```
// we have 8*8 board and 12 pieces (6 for white side and 6 for black)

uint64_t table[12][8][8]; // filled with random values

int position[8][8]; // for each square on board. 0 - no piece. 1..12 - piece

uint64_t hash;

for (int row=0; row<8; row++)
```

[7] https://www.jjj.de/fxt/fxtbook.pdf

```
        for (int col=0; col<8; col++)
        {
                int piece=position[row][col];

                if (piece!=0)
                        hash=hash^table[piece][row][col];
        };

return hash;
```

Now the most interesting part: if the next (modified) chess position differs only by one (moved) piece, you don't need to recalculate hash for the whole position, all you need is:

```
hash=...; // (already calculated)

// subtract information about the old piece:
hash=hash^table[old_piece][old_row][old_col];

// add information about the new piece:
hash=hash^table[new_piece][new_row][new_col];
```

2.1.9 By the way

The usual *OR* also sometimes called *inclusive OR* (or even *IOR*), as opposed to *exclusive OR*. One place is *operator* Python's library: it's called *operator.ior* here.

2.2 AND operation

2.2.1 Checking if a value is on 2^n boundary

If you need to check if your value is divisible by 2^n number (like 1024, 4096, etc.) without remainder, you can use a % operator in C/C++, but there is a simpler way. 4096 is 0x1000, so it always has $4 * 3 = 12$ lower bits cleared.

What you need is just:

```
if (value&0xFFF)
{
        printf ("value is not divisible by 0x1000 (or 4096)\n");
        printf ("by the way, remainder is %d\n", value&0xFFF);
}
else
        printf ("value is divisible by 0x1000 (or 4096)\n");
```

In other words, this code checks if there are any bit set among lower 12 bits. As a side effect, lower 12 bits is always a remainder from division a value by 4096 (because division by 2^n is merely a right shift, and shifted (and dropped) bits are bits of remainder).

Same story if you want to check if the number is odd or even:

```
if (value&1)
        // odd
else
        // even
```

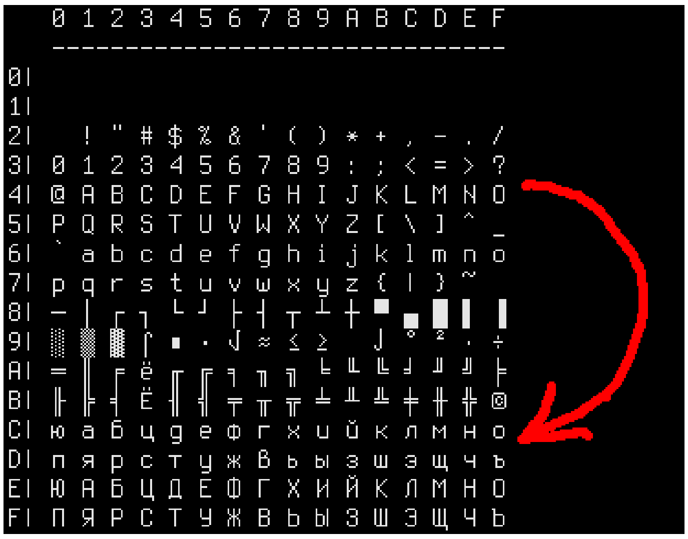
This is merely the same as if to divide by 2 and get 1-bit remainder.

2.2.2 KOI-8R Cyrillic encoding

It was a time when 8-bit ASCII[8] table wasn't supported by some Internet services, including email. Some supported, some others—not.

[8]American Standard Code for Information Interchange

It was also a time, when non-Latin writing systems used second half of 8-bit ASCII table to accommodate non-Latin characters. There were several popular Cyrillic encodings, but KOI-8R (devised by Andrey "ache" Chernov) is somewhat unique in comparison with others.



```
   0 1 2 3 4 5 6 7 8 9 A B C D E F
   -------------------------------
0|
1|
2|   ! " # $ % & ' ( ) * + , - . /
3| 0 1 2 3 4 5 6 7 8 9 : ; < = > ?
4| @ A B C D E F G H I J K L M N O
5| P Q R S T U V W X Y Z [ \ ] ^ _
6| ` a b c d e f g h i j k l m n o
7| p q r s t u v w x y z { | } ~
8| ─ │ ┌ ┐ └ ┘ ├ ┤ ┬ ┴ ┼ ▀ ▄ █ ▌ ▐
9| ░ ▒ ▓ ⌠ ■ ∙ √ ≈ ≤ ≥   ⌡ ° ² · ÷
A| ═ ║ ╒ ё ╓ ╔ ╕ ╖ ╗ ╘ ╙ ╚ ╛ ╜ ╝ ╞
B| ╟ ╠ ╡ Ё ╢ ╣ ╤ ╥ ╦ ╧ ╨ ╩ ╪ ╫ ╬ ©
C| ю а б ц д е ф г х и й к л м н о
D| п я р с т у ж в ь ы з ш э щ ч ъ
E| Ю А Б Ц Д Е Ф Г Х И Й К Л М Н О
F| П Я Р С Т У Ж В Ь Ы З Ш Э Щ Ч Ъ
```

Figure 2.1: KOI8-R table

Someone may notice that Cyrillic characters are allocated almost in the same sequence as Latin ones. This leads to one important property: if all 8th bits in Cyrillic text encoded in KOI-8R are to be reset, a text transforms into transliterated text with Latin characters in place of Cyrillic. For example, Russian sentence:

> Мой дядя самых честных правил, Когда не в шутку занемог, Он уважать себя заставил, И лучше выдумать не мог.

...if encoded in KOI-8R and then 8th bit stripped, transforms into:

> mOJ DQDQ SAMYH ^ESTNYH PRAWIL, kOGDA NE W [UTKU ZANEMOG, oN UWAVATX SEBQ ZASTAWIL, i LU^[E WYDUMATX NE MOG.

...perhaps this is not very appealing æsthetically, but this text is still readable to Russian language natives.

Hence, Cyrillic text encoded in KOI-8R, passed through an old 7-bit service will survive into transliterated, but still readable text.

Stripping 8th bit is automatically transposes any character from the second half of the (any) 8-bit ASCII table to the first one, into the same place (take a look at red arrow right of table). If the character has already been placed in the first half (i.e., it has been in standard 7-bit ASCII table), it's not transposed.

Perhaps, transliterated text is still recoverable, if you'll add 8th bit to the characters which were seems transliterated.

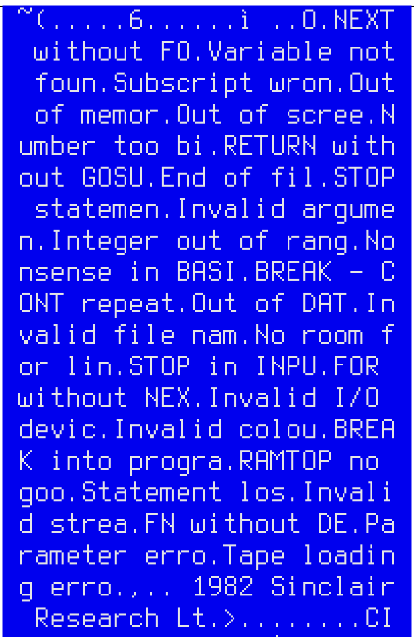
Drawback is obvious: Cyrillic characters allocated in KOI-8R table are not in the same sequence as in Russian/Bulgarian/Ukrainian/etc. alphabet, and this isn't suitable for sorting, for example.

2.3 AND and OR as subtraction and addition

2.3.1 ZX Spectrum ROM text strings

Those who once investigated ZX Spectrum ROM[9] internals, probably noticed that the last symbol of each text string is seemingly absent.

[9]Read-Only Memory



```
~(.....6.......ì ..0.NEXT
 without FO.Variable not
 foun.Subscript wron.Out
 of memor.Out of scree.N
umber too bi.RETURN with
out GOSU.End of fil.STOP
 statemen.Invalid argume
n.Integer out of rang.No
nsense in BASI.BREAK - C
ONT repeat.Out of DAT.In
valid file nam.No room f
or lin.STOP in INPU.FOR
without NEX.Invalid I/O
devic.Invalid colou.BREA
K into progra.RAMTOP no
goo.Statement los.Invali
d strea.FN without DE.Pa
rameter erro.Tape loadin
g erro.,.. 1982 Sinclair
 Research Lt.>........CI
```

Figure 2.2: A part of ZX Spectrum ROM

There are present, in fact.

Here is excerpt of ZX Spectrum 128K ROM disassembled:

```
L048C:  DEFM "MERGE erro"            ; Report 'a'.
        DEFB 'r'+$80
L0497:  DEFM "Wrong file typ"        ; Report 'b'.
        DEFB 'e'+$80
L04A6:  DEFM "CODE erro"             ; Report 'c'.
        DEFB 'r'+$80
L04B0:  DEFM "Too many bracket"      ; Report 'd'.
        DEFB 's'+$80
L04C1:  DEFM "File already exist"    ; Report 'e'.
        DEFB 's'+$80
```

(http://www.matthew-wilson.net/spectrum/rom/128_ROM0.html)

Last character has most significant bit set, which marks string end. Presumably, it was done to save some space? Old 8-bit computers have very tight environment.

Characters of all messages are always in standard 7-bit ASCII table, so it's guaranteed 7th bit is never used for characters.

To print such string, we must check MSB of each byte, and if it's set, we must clear it, then print character, and then stop. Here is a C example:

```
unsigned char hw[]=
{
        'H',
        'e',
        'l',
        'l',
        'o'|0x80
};

void print_string()
{
        for (int i=0; ;i++)
        {
                if (hw[i]&0x80) // check MSB
                {
                        // clear MSB
                        // (in other words, clear all, but leave 7 lower bits intact)
                        printf ("%c", hw[i] & 0x7F);
                        // stop
                        break;
                };
                printf ("%c", hw[i]);
        };
};
```

Now what is interesting, since 7th bit is the most significant bit (in byte), we can check it, set it and remove it using arithmetical operations instead of logical.

I can rewrite my C example:

```
unsigned char hw[]=
{
        'H',
        'e',
        'l',
        'l',
        'o'+0x80
};

void print()
{
        for (int i=0; ;i++)
        {
                // hw[] must have 'unsigned char' type
                if (hw[i] >= 0x80) // check for MSB
                {
                        printf ("%c", hw[i]-0x80); // clear MSB
                        // stop
                        break;
                };
                printf ("%c", hw[i]);
        };
};
```

By default, *char* is signed type in C/C++, so to compare it with variable like 0x80 (which is negative (-128) if treated as signed), we must treat each character in text message as unsigned.

Now if 7th bit is set, the number is always larger or equal to 0x80. If 7th bit is clear, the number is always smaller than 0x80.

Even more than that: if 7th bit is set, it can be cleared by subtracting 0x80, nothing else. If it's not set beforehand, however, subtracting will destruct other bits.

Likewise, if 7th bit is clear, it's possible to set it by adding 0x80. But if it's set beforehand, addition operation will destruct some other bits.

In fact, this is valid for any bit. If the 4th bit is clear, you can set it just by adding 0x10: 0x100+0x10 = 0x110. If the 4th bit is set, you can clear it by subtracting 0x10: 0x1234-0x10 = 0x1224.

It works, because carry isn't happened during addition/subtraction. It will, however, happen, if the bit is already set there before addition, or absent before subtraction.

Likewise, addition/subtraction can be replaced using OR/AND operation if two conditions are met: 1) you want to add/subtract by a number in form of 2^n; 2) this bit in source value is clear/set.

For example, addition of 0x20 is the same as ORing value with 0x20 under condition that this bit is clear before: 0x1204|0x20 = 0x1204+0x20 = 0x1224.

Subtraction of 0x20 is the same as ANDing value with 0x20 (0x....FFDF), but if this bit is set before: 0x1234&(~0x20) = 0x1234&0xFFDF = 0x1234-0x20 = 0x1214.

Again, it works because carry not happened when you add 2^n number and this bit isn't set before.

This property of boolean algebra is important, worth understanding and keeping it in mind.

2.4 Hamming weight / population count

The POPCNT x86 instruction is population count (AKA[10] Hamming weight). It just counts number of bits set in an input value.

As a side effect, the POPCNT instruction (or operation) can be used to determine, if the value has 2^n form. Since, 2^n number always has just one single bit, the POPCNT's result will always be just 1.

For example, I once wrote a base64 strings scanner for hunting something interesting in binary files [11]. And there is a lot of garbage and false positives, so I add an option to filter out data blocks which has size of 2^n bytes (i.e., 256 bytes, 512, 1024, etc.). The size of block is checked just like this:

```
if (popcnt(size)==1)
        // OK
...
```

The instruction is also known as "NSA[12] instruction" due to rumors:

> This branch of cryptography is fast-paced and very politically charged. Most designs are secret; a majority of military encryptions systems in use today are based on LFSRs. In fact, most Cray computers (Cray 1, Cray X-MP, Cray Y-MP) have a rather curious instruction generally known as "population count." It counts the 1 bits in a register and can be used both to efficiently calculate the Hamming distance between two binary words and to implement a vectorized version of a LFSR. I've heard this called the canonical NSA instruction, demanded by almost all computer contracts.

[Bruce Schneier, *Applied Cryptography*, (John Wiley & Sons, 1994)]

[10] Also Known As
[11] https://yurichev.com/news/20210422_base64scanner/
[12] National Security Agency

Chapter 3

Set theory

3.1 Set theory explanation via boolean operations

We are going to put a set in integer variable, where each bit will represent an element of set. Groups of bits - a subset. 'Universe' (all possible elements) is a group of all possible bits within variable.

Cardinality of set is number of elements in it. We can find it just by count all bits or using `popcount()` function (2.4).

3.1.1 Collecting TV series

We're downloading a favorite TV series from torrent tracker. Obviously, they downloading in random order. We want to check if we collected all required movies/elements in 0..7 range.

```
int collected=0;

void add_to_set(int element)
{
        assert(element>=0 && element<=8);
        collected=collect|(1<<element);
        if (collected==0xff)
                printf ("all collected\n");
};
```

Important to note that the `add_to_set()` function can be called several times with the same element.

3.1.2 Rock band

```
#include <assert.h>

#define DRUMS  (1<<0)
#define GUITAR (1<<1)
#define BASS   (1<<2)
#define VOCAL  (1<<3)

// what each rocker can do in our band:
#define JOHN_CAN_PLAY    (GUITAR | BASS)
#define JAKE_CAN_PLAY    (GUITAR | VOCAL)
#define PETER_CAN_PLAY   (GUITAR | DRUMS)
#define MICHAEL_CAN_PLAY (DRUMS | VOCAL)
#define FRANK_CAN_PLAY   (GUITAR | BASS | DRUMS)

// instruments that must be used
// internally, this is 0xf (4 lowest bits)
// this is set universe
#define ROCK_BAND_INSTRUMENTS (GUITAR | BASS | DRUMS | VOCAL)
```

```
// we single out one instrument from each rocker using AND
// then we join them all together using OR, this is 'set union'
// note: we 'take' two instruments from JAKE
// internally, this also must be 0xf (4 lowest bits)
#define STAR_ROCK_BAND  (JAKE_CAN_PLAY & GUITAR) | (JAKE_CAN_PLAY & VOCAL) | (
    JOHN_CAN_PLAY & BASS) | (PETER_CAN_PLAY & DRUMS)

int main()
{
        // this equation must hold:
        assert(STAR_ROCK_BAND==ROCK_BAND_INSTRUMENTS);
};
```

The obvious limitation: only 32 elements in set are allowed in 32-bit integer type. But you can switch to 64-bit integer or to bitstring data type.

3.1.3 Languages

```
#include <assert.h>
#include <stdio.h>

#define ENGLISH (1<<0)
#define FRENCH  (1<<1)
#define GERMAN  (1<<2)
#define SPANISH (1<<3)
#define RUSSIAN (1<<4)
#define ARABIC  (1<<5)
#define CHINESE (1<<6)
// all languages:
#define UNIVERSE (ENGLISH | FRENCH | GERMAN | SPANISH | RUSSIAN | ARABIC | CHINESE)

#define JOHN_CAN_SPEAK          (ENGLISH | FRENCH)
#define JOHN_CAN_UNDERSTAND     (JOHN_CAN_SPEAK | RUSSIAN | GERMAN)

#define PETER_CAN_SPEAK         (ENGLISH | GERMAN)
#define PETER_CAN_UNDERSTAND    (PETER_CAN_SPEAK | SPANISH | ARABIC)

#define NATASHA_CAN_SPEAK       (ENGLISH | RUSSIAN)
#define NATASHA_CAN_UNDERSTAND  (NATASHA_CAN_SPEAK | CHINESE)

#define JOHN_AND_PETER_CAN_UNDERSTAND (JOHN_CAN_UNDERSTAND | PETER_CAN_UNDERSTAND)

int main()
{
        // which languages can JOHN and PETER use to talk with each other?
        // this is 'set intersection'
        assert ((JOHN_CAN_SPEAK & PETER_CAN_SPEAK) == ENGLISH);

        // which languages can Peter speak so that John would understand him?
        // 'set intersection' again
        assert ((PETER_CAN_SPEAK & JOHN_CAN_UNDERSTAND) == (ENGLISH | GERMAN));

        // which languages can John speak so that Peter would understand him?
        assert ((JOHN_CAN_SPEAK & PETER_CAN_UNDERSTAND) == ENGLISH);

        // which languages can John use so that both Peter and Natasha will understand him?
        assert ((JOHN_CAN_SPEAK & (PETER_CAN_UNDERSTAND & NATASHA_CAN_UNDERSTAND)) ==
            ENGLISH);

        // which languages both JOHN and PETER can't speak/understand?
        // this is 'set complement'
        assert (((~JOHN_AND_PETER_CAN_UNDERSTAND) & UNIVERSE) == CHINESE);

        // which languages JOHN can speak that PETER don't?
        // this is 'set subtraction' or 'difference of sets'
        // in other words, clear all bits in JOHN_CAN_SPEAK that present in PETER_CAN_SPEAK
        assert ((JOHN_CAN_SPEAK & (~PETER_CAN_SPEAK)) == FRENCH);

        // which languages PETER can speak that JOHN don't?
        assert ((PETER_CAN_SPEAK & (~JOHN_CAN_SPEAK)) == GERMAN);

        // how many languages our group of 3 persons can speaks?
        // this is 'cardinality' of set
        // we're using a popcount() function here to count all bits in ORed values
        assert (__builtin_popcount (JOHN_CAN_SPEAK | PETER_CAN_SPEAK |
            NATASHA_CAN_SPEAK)==4);

        // which languages can be spoken/understood only by one person in a pair?
        // (a language spoken by both persons is 'eliminated' by XOR operation.)
        assert ((JOHN_CAN_UNDERSTAND ^ PETER_CAN_UNDERSTAND) == (FRENCH | RUSSIAN |
            ARABIC | SPANISH));
};
```

By the way, an interesting story about applying set/logical operations on paper cards:

> To allow a visual check that all cards in a deck were oriented the same way, one corner of each card was beveled, much like Hollerith punched cards. Edge-notched cards, however, were not intended to be read by machines such as IBM card sorters. Instead, they were manipulated by passing one or more slim needles through selected holes in a group of cards. As the needles were lifted, the cards that were notched in the hole positions where the needles were inserted would be left behind as rest of the deck was lifted by the needles. Using two or more needles produced a logical and function. Combining the cards from two different selections produced a logical or. Quite complex manipulations, including sorting were possible using these techniques.

(https://en.wikipedia.org/wiki/Edge-notched_card)

3.1.4 De Morgan's laws

... is valid for both sets, boolean values and bitstrings of arbitrary size.

You can easily prove this for boolean values. Construct a truth table for all possible two boolean inputs. That would consists of only 4 rows. Q.E.D.

By induction, extend this to bitstrings of arbitrary size.

3.1.5 Powerset

(According to Wiktionary: https://en.wiktionary.org/wiki/power_set.) The set whose elements comprise all the subsets of S (including the empty set and S itself). An alternative notation is 2^S...

For example, we have 4 elements in set. Each element for each bit. How can we enumerate all possible combinations? Just by listing all numbers between 0b0000 and 0b1111. That comprises 16 combinations or subsets, including empty set (starting 0b0000).

This is why 2^S notation is also used: there are $2^{numberofelements}$ subsets. In our case, $2^4 = 16$.

3.1.6 Inclusion-exclusion principle

The problem from the "Discrete Structures, Logic and Computability" book by James L. Hein, 4th ed.

```
Suppose a survey revealed that 70% of the population visited
an amusement park and 80% visited a national park.
At least what percentage of the population visited both?
```

The problem is supposed to be solved using finite sets counting and *inclusion–exclusion principle*... But I'm slothful student and would try simple bruteforce.

Each 1 bit in a variable would reflect 10% of population. Then I enumerate all possible pairs. I check only those pairs, where there are 7 bits in p1 variable and 8 bits in p2 variable.

```
#!/usr/bin/env python3

_min=10
_max=0

def popcnt(x):
    #
       https://stackoverflow.com/questions/9829578/fast-way-of-counting-non-zero-bits-in-positive-intege
    return bin(x).count("1")

for p1 in range(2**10): # from 0b0000000000 to 0b1111111111
    if popcnt(p1)!=7:
        continue
    for p2 in range(2**10): # from 0b0000000000 to 0b1111111111
        if popcnt(p2)!=8:
            continue
        # at this point, p1 can take all possible values where sum of bits == 7
        # ... p2 == 8
        x=popcnt (p1 & p2)
        if x==5 or x==7:
            print ("-")
            print ("park1 {0:010b}".format(p1), "(%d)" % popcnt(p1))
            print ("park2 {0:010b}".format(p2), "(%d)" % popcnt(p2))
            print ("both  {0:010b}".format(p1 & p2), "(%d)" % popcnt(p1 & p2))
        # write down min/max values of x:
        _min=min(_min, x)
        _max=max(_max, x)

print ("min:", _min)
print ("max:", _max)
```

The result:

```
min: 5
max: 7
```

You see, the minimum possible value is 5 (50%), the maximum is 7 (70%). Let's add some debug info:

```
...
        if x==5 or x==7:
            print ("-")
            print ("park1 {0:010b}".format(p1), "(%d)" % popcnt(p1))
            print ("park2 {0:010b}".format(p2), "(%d)" % popcnt(p2))
            print ("both  {0:010b}".format(p1 & p2), "(%d)" % popcnt(p1 & p2))
...
```

If maximum:

```
park1 0001111111 (7)
park2 0011111111 (8)
both  0001111111 (7)
```

If minimum:

```
park1 0001111111 (7)
park2 1111111100 (8)
both  0001111100 (5)
```

I've found such a cases, where "ones" are allocated in such a way, so that the AND of "ones" in "both" would be minimal/maximal.

Observing this, we can deduce the general formula: maximal "both" = min(park1, park2)

What about minimal "both"? We can see that "ones" from park1 must "shift out" or "hide in" to a corresponding empty space of park2. So, minimal both = park2 - (100% - park1)

Variations of the problem from the same book:

```
Suppose that 100 senators voted on three separate senate bills as follows:
70 percent of the senators voted for the first bill, 65 percent voted for the second
    bill,
and 60 percent voted for the third bill. At least what percentage of the senators
    voted for all three bills?
```

```
Suppose that 25 people attended a conference with three sessions, where 15 people
    attended the first session,
18 the second session, and 12 the third session. At least how many people attended
    all three sessions?
```

Also, The Abstract Algebra book[1] by I.N. Herstein has exercises like:

```
19. In his book A Tangled Tale, Lewis Carroll proposed the following riddle
about a group of disabled veterans: "Say that 70% have lost an eye, 75%
an ear, 80% an arm, 85% a leg. What percentage, at least, must have lost
all four?" Solve Lewis Carroll's problem.
```

[1]https://archive.org/details/Herstein3thEditon/

Chapter 4

IEEE 754

4.1 IEEE 754 round-off errors

4.1.1 The example from Wikipedia

This fragment from the Wikipedia article has caught my attention:

Listing 4.1: Copypasted from Wikipedia

```
An example of the error caused by floating point roundoff is illustrated in the
    following C code.

int main(){
   double a;
   int i;

   a = 0.2;
   a += 0.1;
   a -= 0.3;

   for (i = 0; a < 1.0; i++)
       a += a;

   printf("i=%d, a=%f\n", i, a);

   return 0;
}

It appears that the program should not terminate. Yet the output is :

i=54, a=1.000000
```

How is that possible?

First of all, I wrote a Wolfram Mathematica code to unpack IEEE 754 single/double values, but with absolute precision (like *bignum*), not bounded by 32 or 64 bits.

Listing 4.2: Wolfram Mathematica code

```
DoubleExpLen = 11;

DoubleMantissaLen = 52;

DoubleBits = DoubleExpLen + DoubleMantissaLen + 1;

DoubleMaxExponent = BitShiftLeft[1, DoubleExpLen - 1] - 1;

DoubleExpMask = BitShiftLeft[1, DoubleExpLen] - 1;
```

```
DoubleMantissaMask = BitShiftLeft[1, DoubleMantissaLen] - 1;

convertDoubleSubnormal[sign_, mantissa_] := (-1)^
   sign*(mantissa/(BitShiftLeft[
      1, ((DoubleBits - (DoubleExpLen + 1)) + DoubleMaxExponent - 1)]))

convertDoubleNaN[sign_, exp_, mantissa_] :=
 If[mantissa == 0 , If[sign == 0, "Inf", "-Inf"], "NaN"]

convertDoubleNormal[sign_, exp_, mantissa_] := (-1)^
   sign*(BitShiftLeft[1, DoubleMantissaLen] +
     mantissa)/(2^(DoubleMantissaLen - (exp - DoubleMaxExponent)))

convertDouble[sign_, exp_, mantissa_] :=
 If[exp == DoubleExpMask, convertDoubleNaN[sign, exp, mantissa],
  If[exp == 0, convertDoubleSubnormal[sign, mantissa],
   convertDoubleNormal[sign, exp, mantissa]]]

convertDoubleQWORD[dw_] :=
 convertDouble[BitShiftRight[dw, DoubleBits - 1],
  BitAnd[BitShiftRight[dw, DoubleMantissaLen], DoubleExpMask],
  BitAnd[dw, DoubleMantissaMask]]
```

The closest value to 0.1 is...

Listing 4.3: Wolfram Mathematica notebook

```
In[]:= N[convertDoubleQWORD[16^^3fb999999999999a], 100]
Out[]= 0.1000000000000000055511151231257827021181583404541015625
```

What are the previous and the next numbers? I decrease/increase mantissa by one bit here:

Listing 4.4: Wolfram Mathematica notebook

```
In[]:= N[convertDoubleQWORD[16^^3fb999999999999a - 1], 100]
Out[]= 0.09999999999999999167332731531132594682276248931884765625

In[]:= N[convertDoubleQWORD[16^^3fb999999999999a + 1], 100]
Out[]= 0.10000000000000001942890293094023945741355419158935546875
```

You see, the first is not exact, but the best approximation possible, in the double-precision IEEE 754 format.

You can't store 0.1 as exact value by the same reason you can't represent $\frac{1}{3}$ value in any binary format. Well, maybe unbalanced ternary computer could, but it will have problems in representing $\frac{1}{2}$ value.

Now I souped up the code from Wikipedia:

```
#include <stdio.h>
#include <stdint.h>
#include <string.h>
#include <math.h>
#include <errno.h>
#include <fenv.h>

const int DoubleExpLen=11;
const int DoubleMantissaLen=52;
const int DoubleBits = DoubleExpLen + DoubleMantissaLen + 1;

void print_double_in_binary (double a)
{
        uint64_t t;
        memcpy (&t, &a, sizeof(double));
        // sign
        printf ("%ld ", (t>>(DoubleBits-1))&1); // sign

        // exponent
        for (int i=(DoubleBits-2); i!=(DoubleBits-2)-DoubleExpLen; i--)
                printf ("%ld", (t>>i)&1);
        printf (" ");

        // mantissa
        for (int i=(DoubleBits-2)-DoubleExpLen; i>=0; i--)
                printf ("%ld", (t>>i)&1);
        printf (" 0x%016lx", t);
};

// https://en.cppreference.com/w/c/numeric/fenv/feround
void show_fe_current_rounding_direction(void)
{
        printf("current rounding direction:  ");
        switch (fegetround()) {
                case FE_TONEAREST:  printf ("FE_TONEAREST");  break;
                case FE_DOWNWARD:   printf ("FE_DOWNWARD");   break;
                case FE_UPWARD:     printf ("FE_UPWARD");     break;
                case FE_TOWARDZERO: printf ("FE_TOWARDZERO"); break;
                default:            printf ("unknown");
        };
        printf("\n");
}

int main(){
        double a;
        int i;

        printf ("0.1: "); print_double_in_binary (0.1); printf ("\n");
        printf ("0.2: "); print_double_in_binary (0.2); printf ("\n");
        printf ("0.3: "); print_double_in_binary (0.3); printf ("\n");
        a = 0.2;
        printf ("a = 0.2;\n");
        printf ("%f %e ", a, a); print_double_in_binary(a); printf ("\n");
        show_fe_current_rounding_direction();
#if 0
        fesetround(FE_DOWNWARD);        // works OK
        fesetround(FE_UPWARD);          // not zero
        fesetround(FE_TONEAREST);       // not zero
        fesetround(FE_TOWARDZERO);      // works OK
#endif
        printf ("a += 0.1;\n");
        feclearexcept(FE_ALL_EXCEPT);
        a += 0.1;
        if(fetestexcept(FE_INEXACT))
                printf ("FE_INEXACT\n");
        printf ("%f %e ", a, a); print_double_in_binary(a); printf ("\n");

        printf ("a -= 0.3;\n");
        feclearexcept(FE_ALL_EXCEPT);
        a -= 0.3;
        if(fetestexcept(FE_INEXACT))
                printf ("FE_INEXACT\n");
        printf ("%f %e ", a, a); print_double_in_binary(a); printf ("\n");

        if (a==0.0)
                printf ("a==0\n");
        else
                printf ("a!=0\n");

        for (i = 0; a < 1.0; i++)
        {
                printf ("i=%d ", i);
                printf ("%f %e ", a, a); print_double_in_binary(a); printf ("\n");
                a += a;
        };

        printf("stop. i=%d, a=%f %e\n", i, a, a);

        return 0;
}
```

Listing 4.5: Output

```
0.1: 0 01111111011 1001100110011001100110011001100110011001100110011010 0x3fb999999999999a
0.2: 0 01111111100 1001100110011001100110011001100110011001100110011010 0x3fc999999999999a
0.3: 0 01111111101 0011001100110011001100110011001100110011001100110011 0x3fd3333333333333
a = 0.2;
0.200000 2.000000e-01 0 01111111100 1001100110011001100110011001100110011001100110011010 0x3fc999999999999a
current rounding direction: FE_TONEAREST
a += 0.1;
FE_INEXACT
0.300000 3.000000e-01 0 01111111101 0011001100110011001100110011001100110011001100110100 0x3fd3333333333334
a -= 0.3;
0.000000 5.551115e-17 0 01111001001 0000000000000000000000000000000000000000000000000000 0x3c90000000000000
a!=0
i=0 0.000000 5.551115e-17 0 01111001001 0000000000000000000000000000000000000000000000000000 0x3c90000000000000
i=1 0.000000 1.110223e-16 0 01111001010 0000000000000000000000000000000000000000000000000000 0x3ca0000000000000
i=2 0.000000 2.220446e-16 0 01111001011 0000000000000000000000000000000000000000000000000000 0x3cb0000000000000
...
i=52 0.250000 2.500000e-01 0 01111111101 0000000000000000000000000000000000000000000000000000 0x3fd0000000000000
i=53 0.500000 5.000000e-01 0 01111111110 0000000000000000000000000000000000000000000000000000 0x3fe0000000000000
stop. i=54, a=1.000000 1.000000e+00
```

After summation of 0.2 and 0.1, FPU reports FE_INEXACT exception in FPU status word. It warns: the result can't "fit" into double IEEE 754 exactly/precisely. The sum in form of 64-bit value is 0x3fd3333333333334. But at the beginning of our program, we dumped 0.3, and it is 0x3fd3333333333333. The difference is one lowest bit of mantissa.

Let's see what Mathematica can say about the exact sum:

Listing 4.6: Wolfram Mathematica notebook

```
In[]:= a = N[convertDoubleQWORD[16^^3fb999999999999a], 100]
Out[]= 0.1000000000000000055511151231257827021181583404541015625

In[]:= b = N[convertDoubleQWORD[16^^3fc999999999999a], 100]
Out[]= 0.200000000000000011102230246251565404236316680908203125

In[]:= a + b
Out[]= 0.3000000000000000166533453693773481063544750213623046875
```

That is the result FPU tries to shove into double-precision IEEE 754 register, but can't.

... while the best approximation of 0.3 is:

Listing 4.7: Wolfram Mathematica notebook

```
In[]:= N[convertDoubleQWORD[16^^3fd3333333333333], 100]
Out[]= 0.299999999999999988897769753748434595763683319091796875
```

And the result produced by our code (lowest bit of mantissa incremented):

Listing 4.8: Wolfram Mathematica notebook

```
In[]:= N[convertDoubleQWORD[16^^3fd3333333333334], 100]
Out[]= 0.3000000000000000444089209850062616169452667236328125
```

Since the rounding mode set by default is FE_TONEAREST, the FPU rounded it to the nearest value.

Now what is that difference by one lowest bit in mantissa?

Listing 4.9: Wolfram Mathematica notebook

```
In[]:= N[convertDoubleQWORD[16^^3fd3333333333334], 100] -
 N[convertDoubleQWORD[16^^3fd3333333333333], 100]

Out[]= 5.5511151231257827021181583404541015625*10^-17
```

This is the 'noise' left in register after subtraction, that isn't equal to zero:

Listing 4.10: Output

```
0.000000 5.551115e-17 0 01111001001 0000000000000000000000000000000000000000000000000000 0x3c90000000000000
a!=0
i=0 0.000000 5.551115e-17 0 01111001001 0000000000000000000000000000000000000000000000000000 0x3c90000000000000
i=1 0.000000 1.110223e-16 0 01111001010 0000000000000000000000000000000000000000000000000000 0x3ca0000000000000
i=2 0.000000 2.220446e-16 0 01111001011 0000000000000000000000000000000000000000000000000000 0x3cb0000000000000
i=3 0.000000 4.440892e-16 0 01111001100 0000000000000000000000000000000000000000000000000000 0x3cc0000000000000
...
```

When that 'noise' gets summed with itself, it grows:

Listing 4.11: Output

```
...
i=49 0.031250 3.125000e-02 0 01111111010 0000000000000000000000000000000000000000000000000000 0x3fa0000000000000
i=50 0.062500 6.250000e-02 0 01111111011 0000000000000000000000000000000000000000000000000000 0x3fb0000000000000
i=51 0.125000 1.250000e-01 0 01111111100 0000000000000000000000000000000000000000000000000000 0x3fc0000000000000
i=52 0.250000 2.500000e-01 0 01111111101 0000000000000000000000000000000000000000000000000000 0x3fd0000000000000
i=53 0.500000 5.000000e-01 0 01111111110 0000000000000000000000000000000000000000000000000000 0x3fe0000000000000
stop. i=54, a=1.000000 1.000000e+00
```

However, you can switch rounding mode to FE_TOWARDZERO:

Listing 4.12: Output

```
a = 0.2;
0.200000 2.000000e-01 0 01111111100 1001100110011001100110011001100110011001100110011010 0x3fc999999999999a
current rounding direction: FE_TONEAREST
a += 0.1;
FE_INEXACT
0.299999 2.999999e-01 0 01111111101 0011001100110011001100110011001100110011001100110011 0x3fd3333333333333
a -= 0.3;
0.000000 0.000000e+00 0 00000000000 0000000000000000000000000000000000000000000000000000 0x0000000000000000
a==0
i=0 0.000000 0.000000e+00 0 00000000000 0000000000000000000000000000000000000000000000000000 0x0000000000000000
i=1 0.000000 0.000000e+00 0 00000000000 0000000000000000000000000000000000000000000000000000 0x0000000000000000
i=2 0.000000 0.000000e+00 0 00000000000 0000000000000000000000000000000000000000000000000000 0x0000000000000000
i=3 0.000000 0.000000e+00 0 00000000000 0000000000000000000000000000000000000000000000000000 0x0000000000000000
...
```

Still FE_INEXACT value, but that works, because rounding gets other direction. The sum is ....33333 (not ....33334). (FE_DOWNWARD would work as well.)

But in a real-world code it's problematic to set rounding mode for each operation...

4.1.2 Conversion from double-precision to single-precision and back

Now this is even more arcane. We add 0.1 to zero and then subtract it, but the result is not zero:

Listing 4.13: Stripped version

```
#include <stdio.h>
#include <stdint.h>
#include <string.h>
#include <math.h>
#include <errno.h>
#include <fenv.h>

int main()
{
        float a;
        //double a; // this will fix it
        int i;

        a = 0.0;
        a += 0.1;
        a -= 0.1;

        if (a==0.0)
                printf ("a==0\n");
        else
                printf ("a!=0\n");

        for (i = 0; a < 1.0; i++)
        {
                printf ("i=%d ", i);
                printf ("%f %e\n", a, a);
                a += a;
        };

        printf("i=%d, a=%f %e\n", i, a, a);

        return 0;
}
```

Listing 4.14: Souped up version

```
#include <stdio.h>
#include <stdint.h>
#include <string.h>
#include <math.h>
#include <errno.h>
#include <fenv.h>

const int FloatExpLen = 8;
const int FloatMantissaLen = 23;
const int FloatBits = FloatExpLen + FloatMantissaLen + 1;

void print_float_in_binary (float a)
{
        uint32_t t;
        memcpy (&t, &a, sizeof(float));

        // sign
        printf ("%d ", (t>>(FloatBits-1))&1);

        // exponent
        for (int i=(FloatBits-2); i!=(FloatBits-2)-FloatExpLen; i--)
                printf ("%d", (t>>i)&1);
        printf (" ");

        // mantissa
        for (int i=(FloatBits-2)-FloatExpLen; i>=0; i--)
                printf ("%d", (t>>i)&1);
        printf (" 0x%08x", t);
};

int main()
{
        float a;
        //double a; // this will fix it
        int i;
        printf ("0.1: "); print_float_in_binary(0.1); printf ("\n");

        a = 0.0;
        printf ("a = 0.0;\n");
        printf ("%f %e ", a, a); print_float_in_binary(a); printf ("\n");
        feclearexcept(FE_ALL_EXCEPT);
        a += 0.1;
        if(fetestexcept(FE_INEXACT))
                printf ("FE_INEXACT\n");
        printf ("a += 0.1;\n");
        printf ("%f %e ", a, a); print_float_in_binary(a); printf ("\n");
        feclearexcept(FE_ALL_EXCEPT);
        a -= 0.1;

        if(fetestexcept(FE_INEXACT))
                printf ("FE_INEXACT\n");
        printf ("a -= 0.1;\n");
        printf ("%f %e ", a, a); print_float_in_binary(a); printf ("\n");

        if (a==0.0)
                printf ("a==0\n");
        else
                printf ("a!=0\n");

        for (i = 0; a < 1.0; i++)
        {
                printf ("i=%d ", i);
                printf ("%f %e ", a, a); print_float_in_binary(a); printf ("\n");
                a += a;
        };

        printf("i=%d, a=%f %e\n", i, a, a);

        return 0;
}
```

Listing 4.15: Output

```
0.1: 0 01111011 10011001100110011001101 0x3dcccccd
a = 0.0;
0.000000 0.000000e+00 0 00000000 00000000000000000000000 0x00000000
FE_INEXACT
a += 0.1;
0.100000 1.000000e-01 0 01111011 10011001100110011001101 0x3dcccccd
FE_INEXACT
a -= 0.1;
0.000000 1.490116e-09 0 01100001 10011001100110011001101 0x30cccccd
a!=0
i=0 0.000000 1.490116e-09 0 01100001 10011001100110011001101 0x30cccccd
i=1 0.000000 2.980232e-09 0 01100010 10011001100110011001101 0x314ccccd
i=2 0.000000 5.960465e-09 0 01100011 10011001100110011001101 0x31cccccd
i=3 0.000000 1.192093e-08 0 01100100 10011001100110011001101 0x324ccccd
...
i=25 0.050000 5.000000e-02 0 01111010 10011001100110011001101 0x3d4ccccd
i=26 0.100000 1.000000e-01 0 01111011 10011001100110011001101 0x3dcccccd
i=27 0.200000 2.000000e-01 0 01111100 10011001100110011001101 0x3e4ccccd
i=28 0.400000 4.000000e-01 0 01111101 10011001100110011001101 0x3ecccccd
i=29 0.800000 8.000000e-01 0 01111110 10011001100110011001101 0x3f4ccccd
i=30, a=1.600000 1.600000e+00
```

How is that possible the 'noise' gets appeared if we just add a constant value and subtract it?

Luckily, I have some assembly knowledge, and I can get into the x86-64 code:

```
...
        cvtss2sd        xmm1, DWORD PTR -4[rbp]
        cvtss2sd        xmm0, DWORD PTR -4[rbp]
        lea     rdi, .LC5[rip]
        mov     eax, 2
        call    printf@PLT
        mov     eax, DWORD PTR -4[rbp]
        movd    xmm0, eax
        call    print_float_in_binary
        mov     edi, 10
        call    putchar@PLT
        mov     edi, 61
        call    feclearexcept@PLT
        cvtss2sd        xmm1, DWORD PTR -4[rbp]
        movsd   xmm0, QWORD PTR .LC6[rip]
        addsd   xmm0, xmm1
        cvtsd2ss        xmm0, xmm0
        movss   DWORD PTR -4[rbp], xmm0
        mov     edi, 32
        call    fetestexcept@PLT
        test    eax, eax
        je      .L8
        lea     rdi, .LC7[rip]
        call    puts@PLT
.L8:
        lea     rdi, .LC8[rip]
        call    puts@PLT
        cvtss2sd        xmm1, DWORD PTR -4[rbp]
        cvtss2sd        xmm0, DWORD PTR -4[rbp]
        lea     rdi, .LC5[rip]
        mov     eax, 2
        call    printf@PLT
        mov     eax, DWORD PTR -4[rbp]
        movd    xmm0, eax
        call    print_float_in_binary
        mov     edi, 10
        call    putchar@PLT
        mov     edi, 61
        call    feclearexcept@PLT
        cvtss2sd        xmm0, DWORD PTR -4[rbp]
        movsd   xmm1, QWORD PTR .LC6[rip]
        subsd   xmm0, xmm1
        cvtsd2ss        xmm0, xmm0
        movss   DWORD PTR -4[rbp], xmm0

...

.LC6:
        .long   2576980378
        .long   1069128089

...
```

But even without that knowledge, we can say something about that code. It uses 'cvtss2sd' and 'cvtsd2ss' instructions, which converts single-precision value to double-precision and back.

These are: "Convert Scalar Single-Precision Floating-Point Value to Scalar Double-Precision Floating-Point Value" and "Convert Scalar Double-Precision Floating-Point Value to Scalar Single-Precision Floating-Point Value".

The 'addsd' and 'subsd' instructions are: "Add Scalar Double-Precision Floating-Point Values" and "Subtract Scalar Double-Precision Floating-Point Value".

(The whole operation is happens in SIMD registers.) Also, the 0.1 constant is encoded as a 64-bit double-precision value (.LC6).

In other words, everything calculates here in double-precision, but stored as single-precision values.

The first FE_INEXACT exception raises when double-precision zero is summed with double-precision 0.1 value and shoved into single-precision register.

```
In[]:= N[convertFloatDWORD[16^^3dcccccd], 100]
Out[]= 0.100000001490116119384765625
```

This is the best possible approximation of 0.1 in single-precision. Now you convert it to double-precision. You'll get something like 0.10000000149011611... But this is not the best possible approximation of 0.1 in double-precision! The best possible approximation is:

```
In[]:= N[convertDoubleQWORD[16^^3fb999999999999a], 100]
Out[]= 0.1000000000000000055511151231257827021181583404541015625
```

You subtract $\approx$ 0.1000000000000000055511151... (best possible approximation) from $\approx$ 0.10000000149011611... and get $\approx$ 0.00000000149011611... Of course, it gets normalized, the mantissa is shifted and exponent aligned, leaving $\approx 1.490116 \cdot 10^{-9}$. After converting this to a single-precision value, it left as the 'noise':

```
...
0.000000 1.490116e-09 0 01100001 10011001100110011001101 0x30cccccd
a!=0
...
```

Again, if summing the 'noise' with itself, it grows.

What if it's compiled for 80x87 FPU support? As it was done before SIMD?

```
% gcc -mfpmath=387 2.c -S -masm=intel -lm

...

        fld     DWORD PTR -8[rbp]
        fld     QWORD PTR .LC8[rip]
        faddp   st(1), st
        fstp    DWORD PTR -8[rbp]

...

        fld     DWORD PTR -8[rbp]
        fld     QWORD PTR .LC8[rip]
        fsubp   st(1), st
        fstp    DWORD PTR -8[rbp]
...


.LC8:
        .long   2576980378
        .long   1069128089
```

You see, one operand for faddp/fsubp instruction (add/sub) loaded as a 32-bit value (single-precision), but another (which is constant) as a 64-bit value (double-precision).

The solution: switching 'float' to 'double' in that code would eliminate problem.

4.1.3 The moral of the story

IEEE 754 is hard. Mathematically flawless expressions and algorithms can go crazy out of the blue.

Wikipedia: Round-off error.

A well-known introduction is What Every ComputerScientist Should Know About Floating-Point Arithmetic by David Goldberg.

The Numerical Recipes book is also recommended, as well as D.Knuth's TAOCP.

The field is numerical analysis.

4.1.4 All the files

(Including the Wolfram Mathematica notebook I used.)

https://αβγ.ελ/current_treeIEEE754/roundoff/files

Chapter 5

Calculus

5.1 What is derivative?

5.1.1 Square

You draw two squares, one has side=10 (units), another has side=11, and can you compare their areas?

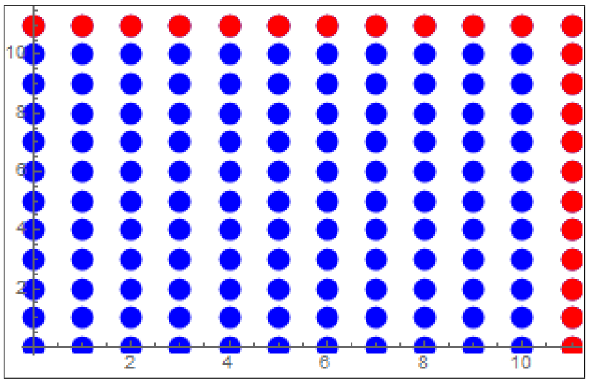
Listing 5.1: Wolfram Mathematica

```
In[]:= square10 = Tuples[Range[0, 10], 2];

In[]:= square11 = Tuples[Range[0, 11], 2];

In[]:= new = Complement[square11, square10];

In[]:= ListPlot[{square11, new}, PlotStyle -> Evaluate[{PointSize[0.04], #} & /@ {
    Blue, Red}]]
```



Red dots are the dots been *added* during growth of the square from 10 to 11. There are 2x of them (or 22 if x=11), and this is exactly the first derivative of x^2:

Listing 5.2: Wolfram Mathematica

```
In[]:= D[x^2, x]
Out[]= 2x
```

Now to get amount of *new dots* that will be *added* if the square growed from 100 to 101, you just calculate $2 \cdot 101 - 1$.

5.1.2 Cube

Now let's see what's with cube.

When it grows, 3 *planes* are added at 3 sides after each *iteration*. Each plane is essentially a square with side of x. Hence, $\approx 3x^2$ "dots" are "added" at each iteration.

And this is indeed the derivative of x^3:

Listing 5.3: Wolfram Mathematica

```
In[]:= D[x^3, x]
Out[]= 3*x^2
```

And what is with tesseract? 4 3D-cubes are *added* at each *iteration*:

Listing 5.4: Wolfram Mathematica

```
In[]:= D[x^4, x]
Out[]= 4*x^3
```

5.1.3 Line

What if your figurine is just a one-dimensional line? It grows by 1 at each iteration:

Listing 5.5: Wolfram Mathematica

```
In[]:= D[x, x]
Out[]= 1
```

5.1.4 Circle

Here I'm generating dots inside of two circles, using Pythagorean theorem. There are circles we have with r=15 and r=16:

Listing 5.6: Wolfram Mathematica

```
In[]:= coords = Tuples[Range[-20, 20], 2];

In[]:= incircle15[coord_] := Abs[coord[[1]]]^2 + Abs[coord[[2]]]^2 <= 15^2

In[]:= circle15 = Select[coords, incircle15];

In[]:= incircle16[coord_] := Abs[coord[[1]]]^2 + Abs[coord[[2]]]^2 <= 16^2

In[]:= circle16 = Select[coords, incircle16];
```

How many dots are added when a circle grows from r=15 to r=16?

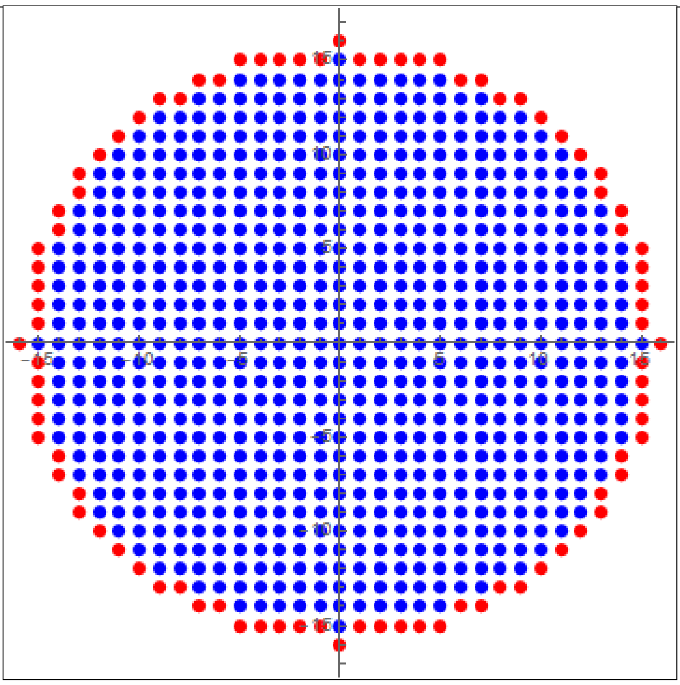
Listing 5.7: Wolfram Mathematica

```
In[]:= new = Complement[circle16, circle15];

In[]:= Length[circle15]
Out[]= 709

In[]:= Length[new]
Out[]= 88

In[]:= ListPlot[{circle15, new}, AspectRatio -> 1,
 PlotStyle -> Evaluate[{PointSize[0.02], #} & /@ {Blue, Red}]]
```



You see, these new dots are like circle circumference in fact.

Indeed, the first derivative of the circle area function is actually its circumference:

Listing 5.8: Wolfram Mathematica

```
In[]:= D[Pi*r^2, r]
Out[]= 2*Pi*r
```

Let's check:

Listing 5.9: Wolfram Mathematica

```
In[]:= Pi*15^2 // N
Out[]= 706.858

In[]:= 2*Pi*15 // N
Out[]= 94.2478
```

Almost equal to amount of dots we generated.

5.1.5 Cylinder

Cylinder is easy. You can say that at each iteration, a new circle is just added at top (or bottom) of it. Indeed, the derivative of cylinder's volume function is just an area of circle with the same radius:

Listing 5.10: Wolfram Mathematica

```
In[]:= D[Pi*r^2*h, h]
Out[]= Pi * r^2
```

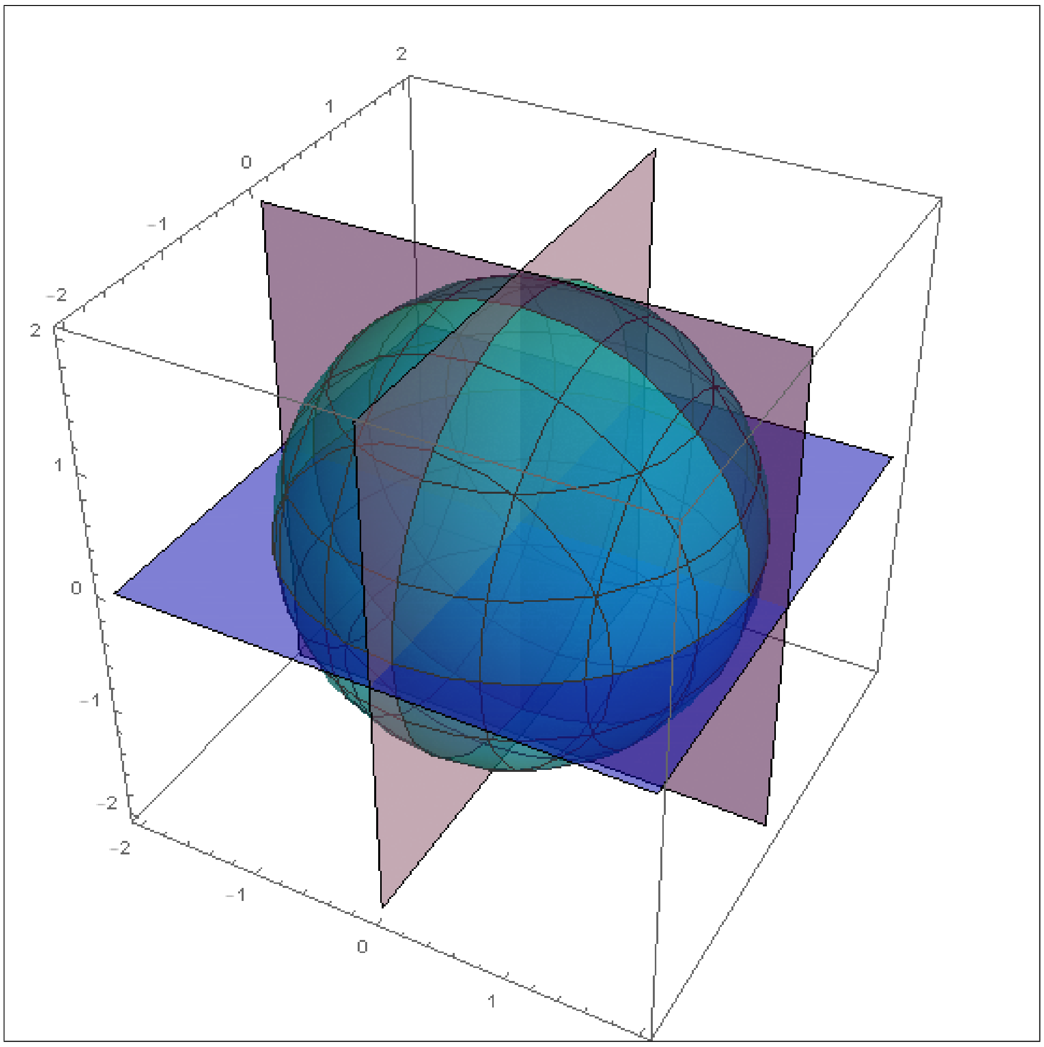
5.1.6 Sphere

When a 3D-sphere grows, you can say that 4 disks with the *new* radius are added into it:

Listing 5.11: Wolfram Mathematica

```
In[]:= D[4/3 (Pi*r^3), r]
Out[]= 4 * Pi * r^2
```

I can show this graphically. Imagine you have a ball and you want to make it bigger. Naive approach is to cut it using knife in 3 planes and insert 3 disks (with the *new* radius) in between, like:



(I got this screenshot from this demonstration)

But each disk must have thickness of 4/3 of one *unit*. Because $\frac{4}{3}3 = 4$.

Sure, new *dots* when added during *growth* are distributed across the whole sphere, but most of them can be placed into these *new* 3 planes.

5.1.7 Conclusion

In other words, derivative is a function that can determine what is added to a function's result, when its input gets incremented by 1.

Chapter 6

Prime numbers

Prime numbers are the numbers which has no divisors except itself and 1. This can be represented graphically.

Let's take 9 balls or some other objects. 9 balls can be arranged into rectangle:

```
ooo
ooo
ooo
```

So are 12 balls:

```
oooo
oooo
oooo
```

Or:

```
ooo
ooo
ooo
ooo
```

So 9 and 12 are not prime numbers. 7 is prime number:

```
ooooooo
```

Or:

```
o
o
o
o
o
o
o
```

It's not possible to form a rectangle using 7 balls, or 11 balls or any other prime number.

The fact that balls can be arranged into rectangle shows that the number can be divided by the number which is represented by height and width of rectangle. Balls of prime number can be arranged vertically or horizontally, meaning, there are only two divisors: 1 and the prime number itself.

6.1 Integer factorization

Natural number can be either prime or composite number. Composite number is a number which can be breaked up by product of prime numbers. Let's take 100. It's not a prime number.

By the *fundamental theorem of arithmetic*, any (composite) number can be represented as a product of prime numbers, in only one single way.

So the *composite number* phrase means that the number is *composed* of prime numbers.

Let's factor 100 in Wolfram Mathematica:

Listing 6.1: Wolfram Mathematica

```
In[]:= FactorInteger[100]
Out[]= {{2, 2}, {5, 2}}
```

This mean that 100 can be constructed using 2 and 5 prime numbers ($2^2 \cdot 5^2$):

Listing 6.2: Wolfram Mathematica

```
In[]:= 2^2*5^2
Out[]= 100
```

You can think about composite number as a rectangle of balls:

```
oooo ... oo
oooo ... oo

...........

oooo ... oo
oooo ... oo
```

You know how many balls are here, but the problem is to get width and/or height.

6.1.1 Using composite number as a container

Even more than that, it's possible to encode some information in prime numbers using factoring. Let's say, we would encode the "Hello" text string. First, let's find ASCII codes of each character in the string:

Listing 6.3: Wolfram Mathematica

```
In[]:= ToCharacterCode["Hello"]
Out[]= {72, 101, 108, 108, 111}
```

Let's find the first 5 prime numbers, each number for each character:

Listing 6.4: Wolfram Mathematica

```
In[]:= Map[Prime[#] &, Range[5]]
Out[]= {2, 3, 5, 7, 11}
```

Build a huge number using prime numbers as bases and ASCII codes as exponents, then get a product of all them ($2^{72} \cdot 3^{101} \cdot 5^{108} \cdot 7^{108} \cdot 11^{111}$):

Listing 6.5: Wolfram Mathematica

```
In[]:= tmp = 2^72*3^101*5^108*7^108*11^111
Out[]= \
1649465578065933994718255257642275679479006861206428830641826551739434\
9344066214616222018844835866267141943107823334187149334898562231349428\
5708281252457614466981636618940129599457183300076472809826225406689893\
5837524252859632074600687844523389231265776082000229507684707641601562\
5000000000000000000000000000000000000000000000000000000000000000000000\
000
```

It's a big number, but Wolfram Mathematica is able to factor it back:

Listing 6.6: Wolfram Mathematica

```
In[]:= FactorInteger[tmp]
Out[]= {{2, 72}, {3, 101}, {5, 108}, {7, 108}, {11, 111}}
```

A first number in each pair is prime number and the second is exponent. Get the text string back:

Listing 6.7: Wolfram Mathematica

```
In[]:= FromCharacterCode[Map[#[[2]] &, tmp]]
Out[]= "Hello"
```

That allows to have some fun. Let's add exclamation point to the end of string by manipulating only the *big number*. ASCII code of exclamation point is 33. The next prime number after 11 is 13. So add it (by multiplying by 13^{33}):

Listing 6.8: Wolfram Mathematica

```
In[]:= tmp = tmp*13^33
Out[]= \
94945390056565777440616156915567505980330247294353321902544691135367 33\
903282354311840549993176158992805279799220663128582267139702321754166 3\
592052181254879362388156851005121497559979376030783799357081813601413 9\
949795868083643018240040555258325648758751938766942671216042126370952 53\
07251454527286114171147346496582031250000000000000000000000000000000\
00000000000000000000000000000000000
```

So we got new number. Let's factor it back and decode:

Listing 6.9: Wolfram Mathematica

```
In[]:= factored = FactorInteger[tmp]
Out[]= {{2, 72}, {3, 101}, {5, 108}, {7, 108}, {11, 111}, {13, 33}}

In[]:= FromCharacterCode[Map[#[[2]] &, factored]]
Out[]= "Hello!"
```

Wow, that works. Will it be possible to remove one 'l' character from the string at the third position? The 'l' has the ASCII code of 108 and it is exponent for two prime numbers in our expression: 5 (first 'l') and 7 (second 'l').

To knock out the character, we divide the *big number* by the corresponding prime number with the exponent of 108 (divide by 5^{108}):

Listing 6.10: Wolfram Mathematica

```
In[]:= tmp = tmp/5^108
Out[]= \
3081154065769189664244341216329094565621009415122099836376732969546063\
1079164051611808432546107410277501678916823138724630810880390384343750\
1196528030610615786507542545262118293483878711112407171889948257893463\
8494741216231004109210436295299274515484540190050751059821909485854359\
9630924207126074604240892753608704

In[]:= factored = FactorInteger[tmp]
Out[]= {{2, 72}, {3, 101}, {7, 108}, {11, 111}, {13, 33}}

In[]:= FromCharacterCode[Map[#[[2]] &, factored]]
Out[]= "Helo!"
```

6.1.2 Using composite number as a container (another example)

Let's say, the initial *container* number is 1. Let's increment the number at the second position within it by multiplying by the first prime number (3):

Listing 6.11: Wolfram Mathematica

```
In[]:= tmp = 1*3
Out[]= 3
```

Then let's set the number at fourth position to 123. The fourth prime number is 7 (the percent sign in Mathematica denotes the last result):

Listing 6.12: Wolfram Mathematica

```
In[]:= tmp = tmp*7^123
Out[]= 2655707111080404050533074341181543827570101833441064348007077 3\
578027976118699964294426564442112809648902 9
```

Then let's set the number at fifth position to 456. The fifth prime number is 11:

Listing 6.13: Wolfram Mathematica

```
In[]:= tmp = tmp*11^456
Out[]= 19917948660639605307938425372554395433764512138284060223646519\
125762196682529345533975108018814491051081332219228828716217649997680 0\
91470685911607982433085918832946490693550155584725644574228290739381 18\
43962049990518568799401019343399136009424510067479822915249106531850 84\
40579728965378943012137352524186397829745920773930283901930601389365 03\
01254655789585673776270638156202615579394840366285362309661582229607 62\
85098996415744775476581424575981684970063096085998305547589516724845 33\
92161058637544637129571437325633759901989010736986265849031640278504 51\
882565983711408058957308 7269
```

Then let's decrement the number at fourth position, the fourth prime number is 7:

Listing 6.14: Wolfram Mathematica

```
In[]:= tmp = tmp/7
Out[]= 2845421237234229329705489338936342204823501734040580031949502 7\
3225174238321847793342501543125921300729733317417554695945966428538287\
0210097987372568919012274118992355813364307940675092082032612962768740\
6280292855788366971343002763342733715632072866782831845035586647407263\
4368532709339849001733907503455199689963702967704326271704371627052147\
1607807969940810539467234022314659368484977195183623187094511747086804\
0728428059392110782368774939425954995723299440856900792512788103549334\
1737294091077805304224491046519108557427001533855180835575948611214931\
26080854815915436993901246 7
```

Let's factor the composite number and get all the numbers we set inside *container* (1, 122, 456):

Listing 6.15: Wolfram Mathematica

```
In[]:= FactorInteger[tmp]
Out[]= {{3, 1}, {7, 122}, {11, 456}}
```

So the resulting number has $3 \cdot 7^{122} \cdot 11^{456}$ form at the end.

This is somewhat wasteful way to store the numbers, but out of curiosity: since there are infinite number of prime numbers and so any number of infinitely big numbers can be *stored* within one (although huge) composite number.

6.2 Coprime numbers

Coprime numbers are the 2 or more numbers which don't have any common divisors. In mathematical lingo, the GCD of all coprime numbers is 1.

3 and 5 are coprimes. So are 7 and 10. So are 4, 5 and 9.

Coprime numbers are the numerator and denominator in fraction which cannot be reduced further (irreducible fraction). For example, $\frac{130}{14}$ is $\frac{65}{7}$ after reduction (or simplification), 65 and 7 are coprime to each other, but 130 and 14 are not (they has 2 as common divisor).

One application of coprime numbers in engineering is to make number of cogs on cogwheel and number of chain elements on chain to be coprimes. Let's imagine bike cogwheels and chain:

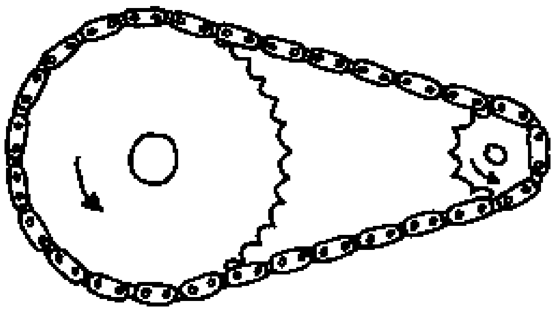


Figure 6.1: The picture was taken from www.mechanical-toys.com

If you choose 5 as number of cogs on one of cogwhell and you have 10 or 15 or 20 chain elements, each cog on cogwheel will meet some set of the same chain elements. For example, if there are 5 cogs on cogwheel and 20 chain elements, each cog will meet only 4 chain elements and vice versa: each chain element has its *own* cog on cogwheel. This is bad because both cogwheel and chain will run out slightly faster than if each cog would interlock every chain elements at some point. To reach this, number of cogs and chain elements could be coprime numbers, like 5 and 11, or 5 and 23. That will make each chain element interlock each cog evenly, which is better.

6.3 Semiprime numbers

Semiprime is a product of two prime numbers.

An interesting property of semiprime:

```
In 1974 the Arecibo message was sent with a radio signal aimed at a star cluster. It
    consisted of 1679 binary digits intended to be interpreted as a 23×73 bitmap image
    . The number 1679 = 23×73 was chosen because it is a semiprime and therefore can
    only be broken down into 23 rows and 73 columns, or 73 rows and 23 columns.
```

(https://en.wikipedia.org/wiki/Semiprime)

6.4 How RSA works

RSA public-key cryptosystem (named after its inventors: Ron Rivest, Adi Shamir and Leonard Adleman) is the most used asymmetric public-private key cryptosystem.

6.4.1 Fermat little theorem

Fermat little theorem states that if p is prime, this congruence is valid for any a in the environment of modulo arithmetic of base p:

$$a^{p-1} \equiv 1 \pmod p$$

There are proofs, which are, perhaps, too tricky for this article which is intended for beginners. So far, you can just take it as granted.

Here is one important property of this theorem. The expression a^{p-1} will always return 1 modulo p for *any* a and *any* prime p.

The following Python code would always work:

```
#!/usr/bin/env python3
import random

# https://stackoverflow.com/questions/567222/simple-prime-number-generator-in-python
def is_prime(num):
    """Returns True if the number is prime else False."""
    if num == 0 or num == 1:
        return False
    for x in range(2, num):
        if num % x == 0:
            return False
    else:
        return True

primes=[i for i in range(1, 10000) if is_prime(i)]
# random prime:
p=random.choice(primes)
# random x:
x=random.randint(1, 10*10)
assert pow(x, p-1, p)==1
```

Wolfram Mathematica can even reduce it:

Listing 6.16: Wolfram Mathematica

```
In[]:= PowerMod[x, (p-1), p]
Out[]= 1

In[]:= Mod[x^(p-1), p]
Out[]= 1
```

This theorem can be used to sieve prime numbers. So you take, for example, 10 and test it. Let's take some random a value (123) (Wolfram Mathematica):

Listing 6.17: Wolfram Mathematica

```
In[]:= Mod[123^(10 - 1), 10]
Out[]= 3
```

We've got 3, which is not 1, indicating the 10 is not prime. On the other hand, 11 is prime:

Listing 6.18: Wolfram Mathematica

```
In[]:= Mod[123^(11 - 1), 11]
Out[]= 1
```

This method is not perfect, some composite p numbers can lead to 1, for example p=1105, but can be used as a method to sieve vast amount of prime numbers candidates.

6.4.2 Euler's totient function

It is a number of coprime numbers under some n. Denoted as $\phi(n)$, pronounced as *phi*. For the sake of simplification, you may just keep in mind that if $n = pq$ (i.e., product of two prime numbers), $\varphi(pq) = (p - 1)(q - 1)$. This is true for RSA environment.

6.4.3 Euler's theorem

Euler's theorem is a generalization of Fermat little theorem. It states:

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

But again, for the sake of simplification, we may keep in mind that Euler's theorem in the RSA environment is this:

$$a^{(p-1)(q-1)} \equiv 1 \pmod{n}$$

... where $n = pq$ and both p and q are prime numbers.

This theorem is central to RSA algorithm.

6.4.4 RSA example

There are *The Sender* and *The Receiver*. *The Receiver* generates two big prime numbers (p and q) and publishes its product ($n = pq$). Both p and q are kept secret.

For the illustration, let's randomly pick p and q among the first 50 prime numbers in Wolfram Mathematica:

Listing 6.19: Wolfram Mathematica

```
In[]:= p = Prime[RandomInteger[50]]
Out[]= 89

In[]:= q = Prime[RandomInteger[50]]
Out[]= 43

In[]:= n = p*q
Out[]= 3827
```

3827 is published as public key, named *public key modulus* or *modulo*. It is semiprime. There is also public key exponent e, which is not secret, is often 65537, but we will use 17 to keep all results tiny.

Now *The Sender* wants to send a message (123 number) to *The Receiver* and he/she uses one-way function:

Listing 6.20: Wolfram Mathematica

```
In[]:= e = 17
Out[]= 17

In[]:= encrypted = Mod[123^e, n]
Out[]= 3060
```

3060 is encrypted message, which can be decrypted only using p and q values separately. This is one-way function, because only a part of exponentiation result is left. One and important consequence is that even *The Sender* can't decrypt it. This is why you can encrypt a piece of text in PGP/GnuPG to someone using his/her public key, but can't decrypt it. Perhaps, that's how CryptoLockers works, making impossible to decrypt the files.

To recover message (123), p and q values must be known.

First, we get the result of Euler's totient function $(p-1)(q-1)$ (this is the point where p and q values are needed):

Listing 6.21: Wolfram Mathematica

```
In[]:= totient = (p - 1)*(q - 1)
Out[]= 3696
```

Now we calculating decrypting exponent using multiplicative modulo inverse (multiplicative inverse was also described in here (7) ($e^{-1} \pmod{totient = (p-q)(q-1)}$)):

Listing 6.22: Wolfram Mathematica

```
In[]:= d = PowerMod[e, -1, totient]
Out[]= 2609
```

Now decrypt the message:

Listing 6.23: Wolfram Mathematica

```
In[18]:= Mod[encrypted^d, n]
Out[18]= 123
```

So the d exponent forms another one-way function, restoring the work of what was done during encryption.

6.4.5 So how it works?

It works, because e and d exponents are reciprocal to each other by modulo $totient = (p-1)(q-1)$:

Listing 6.24: Wolfram Mathematica

```
In[]:= Mod[e*d, totient] (* check *)
Out[]= 1
```

This allows...

$$m^{ed} = m \pmod{n} \tag{6.1}$$

Or in Mathematica:

Listing 6.25: Wolfram Mathematica

```
In[]:= Mod[123^(e*d), n]
Out[]= 123
```

So the encryption process is $m^e \pmod{n}$, decryption is $(m^e)^d = m \pmod{n}$.

To prove congruence 6.1, we first should prove the following congruence:

$$ed \equiv 1 \pmod{((p-1)(q-1))}$$

... using modular arithmetic rules, it can be rewritten as:

$$ed = 1 + h(p-1)(q-1)$$

h is some unknown number which is present here because it's not known how many times the final result was *rounded* while exponentiation (this is modulo arithmetic after all).

So $m^{ed} = m \pmod{n}$ can be rewritten as:

$$m^{1+h((p-1)(q-1))} \equiv m \pmod{n}$$

...and then to:

$$m\left(m^{(p-1)(q-1)}\right)^h \equiv m \pmod{n}$$

The last expression can be simplified using Euler's theorem (stating that $a^{(p-1)(q-1)} \equiv 1 \pmod{n}$). The result is:

$$m(1)^h \equiv m \pmod{n}$$

... or just:

$$m \equiv m \pmod{n}$$

6.4.6 Breaking RSA

We can try to factor n semiprime (or RSA modulus) in Mathematica:

Listing 6.26: Wolfram Mathematica

```
In[]:= FactorInteger[n]
Out[]= {{43, 1}, {89, 1}}
```

And we getting correct p and q, but this is possible only for small values. When you use some big ones, factorizing is extremely slow, making RSA unbreakable, if implemented correctly.

The bigger p, q and n numbers, the harder to factorize n, so the bigger keys in bits are, the harder it to break.

6.4.7 The difference between my simplified example and a real RSA algorithm

In my example, public key is $n = pq$ (product) and secret key are p and q values stored separately. This is not very efficient, to calculate totient and decrypting exponent each time. So in practice, a public key is n and e, and a secret key is at least n and d, and d is stored in secret key precomputed.

For example, here is my PGP public key[1]:

```
dennis@...:~$ gpg --export-options export-reset-subkey-passwd --export-secret-subkeys
    0x3B262349\! | pgpdump
Old: Secret Key Packet(tag 5)(533 bytes)
        Ver 4 - new
        Public key creation time - Tue Jun 30 02:08:38 EEST 2015
        Pub alg - RSA Encrypt or Sign(pub 1)
        RSA n(4096 bits) - ...
        RSA e(17 bits) - ...
...
```

[1] https://yurichev.com/pgp.html

... so there are available openly big (4096 bits) n and e (17 bits).

And here is my PGP secret key:

```
dennis@...:~$ gpg --export-options export-reset-subkey-passwd --export-secret-subkeys
    0x55B5C64F\! | pgpdump
gpg: about to export an unprotected subkey

You need a passphrase to unlock the secret key for
user: "Dennis Yurichev <dennis@yurichev.com>"
4096-bit RSA key, ID 55B5C64F, created 2015-06-29

gpg: gpg-agent is not available in this session
Old: Secret Key Packet(tag 5)(533 bytes)
        Ver 4 - new
        Public key creation time - Tue Jun 30 02:08:38 EEST 2015
        Pub alg - RSA Encrypt or Sign(pub 1)
        RSA n(4096 bits) - ...
        RSA e(17 bits) - ...
...
Old: Secret Subkey Packet(tag 7)(1816 bytes)
        Ver 4 - new
        Public key creation time - Tue Jun 30 02:08:38 EEST 2015
        Pub alg - RSA Encrypt or Sign(pub 1)
        RSA n(4096 bits) - ...
        RSA e(17 bits) - ...
        RSA d(4093 bits) - ...
        RSA p(2048 bits) - ...
        RSA q(2048 bits) - ...
        RSA u(2048 bits) - ...
        Checksum - 94 53
...
```

... it has all variables stored in the file, including d, p and q.

6.4.8 A real example: OpenSSL

Listing 6.27: Linux console

```
# Generating a 32-bit RSA key pair:
$ openssl genrsa -out keypair.pem 32

# Dump the private key:
$ openssl rsa -noout -text -in keypair.pem

# Extract the public key from private:
$ openssl rsa -in keypair.pem -pubout -out pub.pub

# Dump the public key:
$ openssl rsa -noout -text -inform PEM -pubin -in pub.pub
```

N.B.: your build of OpenSSL may deny to generate too small keys[2].

Listing 6.28: The private key

```
$ openssl rsa -noout -text -in keypair.pem

RSA Private-Key: (32 bit, 2 primes)
modulus: 3231798799 (0xc0a1560f)
publicExponent: 65537 (0x10001)
privateExponent: 1892254033 (0x70c98151)
prime1: 64109 (0xfa6d)
```

[2]https://superuser.com/questions/1640182/why-am-i-getting-an-error-when-trying-to-generate-rsa-128

```
prime2: 50411 (0xc4eb)
exponent1: 42305 (0xa541)
exponent2: 13863 (0x3627)
coefficient: 29340 (0x729c)
```

Listing 6.29: The public key

```
$ openssl rsa -noout -text -inform PEM -pubin -in pub.pub

RSA Public-Key: (32 bit)
Modulus: 3231798799 (0xc0a1560f)
Exponent: 65537 (0x10001)
```

The private key also have many precomputed values. All they are described in RFC 3447[3]. But the most important values are two primes, of course.

Let's check them all in Wolfram Mathematica:

Listing 6.30: Wolfram Mathematica

```
In[]:= prime1=p=64109
Out[]= 64109

In[]:= prime2=q=50411
Out[]= 50411

In[]:= modulus=n=p*q
Out[]= 3231798799

In[]:= lambdaN=LCM[p-1,q-1] (* not used. Phi is used instead: (p-1)(q-1) *)
Out[]= 1615842140

In[]:= exponent=privateExponent=d=65537
Out[]= 65537

In[]:= privateExponent=e=PowerMod[d,-1,(p-1)(q-1)]
Out[]= 1892254033

In[]:= exponent1=PowerMod[d,-1,p-1]
Out[]= 42305

In[]:= exponent2=PowerMod[d,-1,q-1]
Out[]= 13863

In[]:= coefficient=PowerMod[q,-1,p]
Out[]= 29340
```

Recovering all these values from the public key is a question of factoring modulus:

Listing 6.31: Wolfram Mathematica

```
In[]:= FactorInteger[modulus]
Out[]= {{50411, 1}, {64109, 1}}
```

When dumping bigger keys, OpenSSL prints long hexadecimal numbers, like:

```
RSA Public-Key: (256 bit)
Modulus:
    00:c7:b5:dc:5c:b0:73:bd:1d:b2:96:bb:49:26:a6:
    82:67:89:77:21:5e:39:54:89:a8:d1:d5:89:af:5a:
    34:06:ed
Exponent: 65537 (0x10001)
```

[3] https://datatracker.ietf.org/doc/html/rfc3447

Just remove colons (':') and join all the numbers, so the Modulus here is:
0x00c7b5dc5cb073bd1db296bb4926a682678977215e395489a8d1d589af5a3406ed.

6.4.9 The RSA signature

It's possible to sign a message to publish it, so everyone can check the signature. For example, *The Publisher* wants to sign the message (let's say, 456). Then he/she uses d exponent to compute signature:

Listing 6.32: Wolfram Mathematica

```
In[]:= signed = Mod[456^d, n]
Out[]= 2282
```

Now he publishes $n = pq$ (3827), e (17 in our example), the message (456) and the signature (2282). Some other *Consumers* can verify its signature using e exponent and n:

Listing 6.33: Wolfram Mathematica

```
In[]:= Mod[2282^e, n]
Out[]= 456
```

... this is another illustration that e and d exponents are complement each other, by modulo $totient = (p-1)(q-1)$.

The signature can only be generated with the access to p and q values, but it can be verified using product ($n = pq$) value.

6.4.10 Hybrid cryptosystem

RSA is slow, because exponentiation is slow and exponentiation by modulo is also slow. Perhaps, this is the reason why it was considered impractical by GCHQ when Clifford Cocks first came with this idea in 1970s. So in practice, if *The Sender* wants to encrypt some big piece of data to *The Receiver*, a random number is generated, which is used as a key for symmetrical cryptosystem like DES or AES. The piece of data is encrypted by the random key. The key is then encrypted by RSA to *The Receiver* and destroyed. *The Receiver* recovers the random key using RSA and decrypts all the data using DES or AES.

Chapter 7

Modulo arithmetics

7.1 Quick introduction into modular arithmetic

Instead of epigraph:

> One of the earliest practical applications of number theory occurs in gears. If two cogs are placed together so that their teeth mesh, and one cog has m teeth, the other n teeth, then the movement of the cogs is related to these numbers. For instance, suppose one cog has 30 teeth and the other has 7. If I turn the big cog exactly once, what does the smaller one do? It returns to the initial position after 7, 14, 21 and 28 steps. So, the final 2 steps, to make 30, advance it by just 2 steps. This number turns up because it is the remainder on dividing 30 by 7. So the motion of cogs is a mechanical representation of division with remainder, and this is the basis of modular arithmetic.

(Ian Stewart – Taming the Infinite: The Story of Mathematics from the First Numbers to Chaos Theory)

Modular arithmetic is an environment where all values are limited by some number (modulo). Many textbooks has clock as example. Let's imagine old mechanical analog clock. There hour hand points to one of number in bounds of 0..11 (zero is usually shown as 12). What hour will be if to sum up 10 hours (no matter, AM or PM) and 4 hours? 10+4 is 14 or 2 by modulo 12. Naively you can just sum up numbers and subtract modulo base (12) as long as it's possible.

Modern digital watch shows time in 24 hours format, so hour there is a variable in modulo base 24. But minutes and seconds are in modulo 60 (let's forget about leap seconds for now).

Another example is US imperial system of measurement: human height is measured in feets and inches. There are 12 inches in feet, so when you sum up some lengths, you increase feet variable each time you've got more than 12 inches.

Another example I would recall is password cracking utilities. Often, characters set is defined in such utilities. And when you set all Latin characters plus numbers, you've got 26+10=36 characters in total. If you brute-forcing a 6-characters password, you've got 6 variables, each is limited by 36. And when you increase last variable, it happens in modular arithmetic rules: if you got 36, set last variable to 0 and increase penultimate one. If it's also 36, do the same. If the very first variable is 36, then stop. Modular arithmetic may be very helpful when you write multi-threading (or distributed) password cracking utility and you need to slice all passwords space by even parts.

This is yet another application of modulo arithmetic, which many of us encountered in childhood.

Given a counting rhyme, like:

```
    Eeny, meeny, miny, moe,
    Catch a tiger by the toe.
    If he hollers, let him go,
    Eeny, meeny, miny, moe.
```

(https://en.wikipedia.org/wiki/Eeny,_meeny,_miny,_moe)

... predict, who will be chosen.

If I'm correct, that rhyme has 16 items, and if a group of kids consist of, say, 5 kids, who will be chosen? 16 mod 5 = 1, meaning, the next kid after the one at whom counting had begun.

Or 7 kids, 16 mod 7 = 2. Count two kids after the first one.

If you can calculate this quickly, you can probably get an advantage by choosing a better place within a circle...

Now let's recall old mechanical counters which were widespread in pre-digital era:

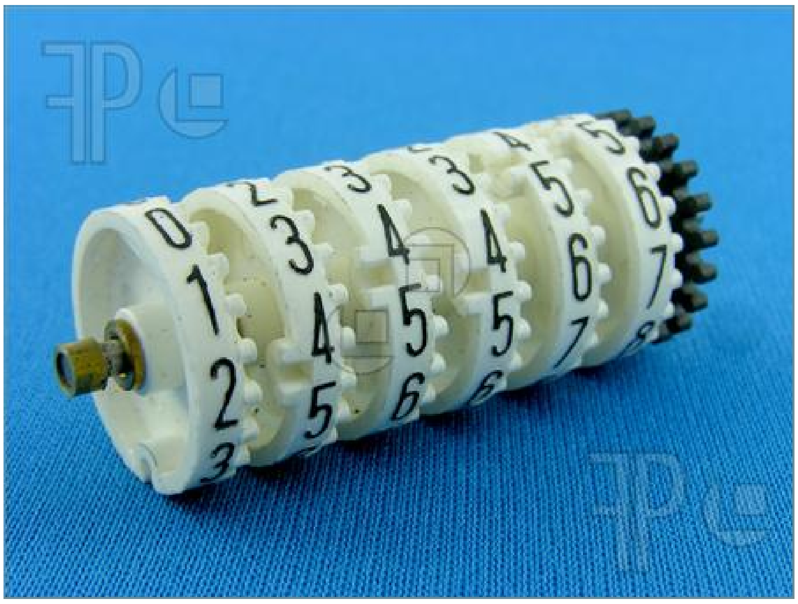


Figure 7.1: The picture was stolen from http://www.featurepics.com/ — sorry for that!

This counter has 6 wheels, so it can count from 0 to $10^6 - 1$ or 999999. When you have 999999 and you increase the counter, it will resetting to 000000— this situation is usually understood by engineers and computer programmers as overflow. And if you have 000000 and you decrease it, the counter will show you 999999. This situation is often called "wrap around". See also: http://en.wikipedia.org/wiki/Integer_overflow.

7.1.1 Modular arithmetic on CPUs

The reason I talk about mechanical counter is that CPU registers acting in the very same way, because this is, perhaps, simplest possible and efficient way to compute using integer numbers.

This implies that almost all operations on integer values on your CPU is happens by modulo 2^{32} or 2^{64} depending on your CPU. For example, you can sum up 0x87654321 and 0xDEADBABA, which resulting in 0x16612FDDB. This value is too big for 32-bit register, so only 0x6612FDDB is stored, and leading 1 is dropped. If you will multiply these two numbers, the actual result it 0x75C5B266EDA5BFFA, which is also too big, so only low 32-bit part is stored into destination register: 0xEDA5BFFA. This is what happens when you multiply numbers in plain C/C++ language, but some readers may argue: when sum is too big for register, CF (carry flag) is set, and it can be used after. And there is x86 MUL instruction which in fact produces 64-bit result in 32-bit environment (in EDX:EAX registers pair). That's true, but observing just 32-bit registers, this is exactly environment of modulo with base 2^{32}.

Now that leads to a surprising consequence: almost every result of arithmetic operation stored in general purpose register of 32-bit CPU is in fact remainder of division operation: result is always divided by 2^{32} and remainder is left in register. For example, 0x16612FDDB is too large for storage, and it's divided by 2^{32} (or 0x100000000). The result of division (quotient) is 1 (which is dropped) and remainder is 0x6612FDDB (which is stored as a result). 0x75C5B266EDA5BFFA divided by 2^{32} (0x100000000) produces 0x75C5B266 as a result of division (quotient) and 0xEDA5BFFA as a remainder, the latter is stored.

And if your code is 32-bit one in 64-bit environment, CPU registers are bigger, so the whole result can be stored there, but high half is hidden behind the scenes – because no 32-bit code can access it.

By the way, this is the reason why remainder calculation is often called ”division by modulo”. C/C++ has a percent sign (“%”) for this operation, but some other PLs like Pascal and Haskell has ”mod” operator.

Usually, almost all sane computer programmers works with variables as they never wrapping around and value here is always in some limits which are defined preliminarily. However, this implicit division operation or ”wrapping around” can be exploited usefully.

7.1.2 Getting random numbers

When you write some kind of videogame, you need random numbers, and the standard C/C++ rand() function gives you them in 0..0x7FFF range (MSVC) or in 0..0x7FFFFFFF range (GCC). And when you need a random number in 0..10 range, the common way to do it is:

```
X_coord_of_something_spawned_somewhere=rand() % 10;
Y_coord_of_something_spawned_somewhere=rand() % 10;
```

No matter what compiler do you use, you can think about it as 10 is subtracted from rand() result, as long as there is still a number bigger than 10. Hence, result is remainder of division of rand() result by 10.

One nasty consequence is that neither 0x8000 nor 0x80000000 cannot be divided by 10 evenly, so you'll get some numbers slightly more often than others.

I tried to calculate in Mathematica. Here is what you get if you write <i>rand()

```
In[]:= Counts[Map[Mod[#, 3] &, Range[0, 16^^8000 - 1]]]
Out[]= <|0 -> 10923, 1 -> 10923, 2 -> 10922|>
```

So a number 2 appers slightly seldom than others.

Here is a result for *rand() % 10*:

```
In[]:= Counts[Map[Mod[#, 10] &, Range[0, 16^^8000 - 1]]]
Out[]= <|0 -> 3277, 1 -> 3277, 2 -> 3277, 3 -> 3277, 4 -> 3277,
 5 -> 3277, 6 -> 3277, 7 -> 3277, 8 -> 3276, 9 -> 3276|>
```

Numbers 8 and 9 appears slightly seldom (3276 against 3277).

Here is a result for *rand() % 100*:

```
In[]:= Counts[Map[Mod[#, 100] &, Range[0, 16^^8000 - 1]]]
Out[]= <|0 -> 328, 1 -> 328, 2 -> 328, 3 -> 328, 4 -> 328, 5 -> 328,
  6 -> 328, 7 -> 328, 8 -> 328, 9 -> 328, 10 -> 328, 11 -> 328,
 12 -> 328, 13 -> 328, 14 -> 328, 15 -> 328, 16 -> 328, 17 -> 328,
 18 -> 328, 19 -> 328, 20 -> 328, 21 -> 328, 22 -> 328, 23 -> 328,
 24 -> 328, 25 -> 328, 26 -> 328, 27 -> 328, 28 -> 328, 29 -> 328,
 30 -> 328, 31 -> 328, 32 -> 328, 33 -> 328, 34 -> 328, 35 -> 328,
 36 -> 328, 37 -> 328, 38 -> 328, 39 -> 328, 40 -> 328, 41 -> 328,
 42 -> 328, 43 -> 328, 44 -> 328, 45 -> 328, 46 -> 328, 47 -> 328,
 48 -> 328, 49 -> 328, 50 -> 328, 51 -> 328, 52 -> 328, 53 -> 328,
 54 -> 328, 55 -> 328, 56 -> 328, 57 -> 328, 58 -> 328, 59 -> 328,
 60 -> 328, 61 -> 328, 62 -> 328, 63 -> 328, 64 -> 328, 65 -> 328,
 66 -> 328, 67 -> 328, 68 -> 327, 69 -> 327, 70 -> 327, 71 -> 327,
 72 -> 327, 73 -> 327, 74 -> 327, 75 -> 327, 76 -> 327, 77 -> 327,
 78 -> 327, 79 -> 327, 80 -> 327, 81 -> 327, 82 -> 327, 83 -> 327,
 84 -> 327, 85 -> 327, 86 -> 327, 87 -> 327, 88 -> 327, 89 -> 327,
 90 -> 327, 91 -> 327, 92 -> 327, 93 -> 327, 94 -> 327, 95 -> 327,
 96 -> 327, 97 -> 327, 98 -> 327, 99 -> 327|>
```

...now larger part of numbers happens slightly seldom, these are 68...99.

This is sometimes called *modulo bias*. It's perhaps acceptable for videogames, but may be critical for scientific simulations, including Monte Carlo method.

Constructing a PRNG[1] with uniform distribution may be tricky, there are couple of methods: http://www.reddit.com/r/algorithms/comments/39tire/using_a_01_generator_generate_a_random_number/, http://www.prismmodelchecker.org/casestudies/dice.php.

7.1.3 Reverse Engineering

Not uncommon in 32-bit x86 code, you may find an instruction like:

Listing 7.1: 32-bit x86 code

```
add ecx, 0FFFFFFCDh
```

It adds 0xFFFFFFCD to a value in a register (or variable). But why? What for?

Let's find out in Wolfram Mathematica:

Listing 7.2: Wolfram Mathematica

```
In[]:= 16^^FFFFFFCD
Out[]= 4294967245

In[]:= Mod[10000 + 4294967245, 2^32]
Out[]= 9949
```

In other words, this instruction *subtracts* 51 from a variable! Somehow, a compiler (MSVC in this case) decided that addition operation would be better (or faster?) than subtraction operation. I'm not sure why a compiler can this, but MSVC do this a lot. Anyway, this is subtraction, and these operations are equivalent to each other.

It's like adding 0xffffffff which is equivalent to subtracting 1. However, subtracting 0xffffffff is equivalent to adding 1.

Same story for a simple Caesar cipher: you may even don't know what operation was used during encryption: addition or subtraction. By cryptanalysis you can find two keys: for addition and subtraction.

7.2 Remainder of division

7.2.1 Remainder of division by modulo 2^n

... can be easily computed with AND operation. If you need a random number in range of 0..16, here you go: rand()&0xF. That helps sometimes.

For example, you need a some kind of wrapping counter variable which always should be in 0..16 range. What you do? Programmers often write this:

```
int counter=0;
...
counter++;
if (counter==16)
    counter=0;
```

But here is a version without conditional branching:

```
int counter=0;
...
counter++;
counter=counter&0xF;
```

As an example, this I found in the git source code:

```
char *sha1_to_hex(const unsigned char *sha1)
{
        static int bufno;
        static char hexbuffer[4][GIT_SHA1_HEXSZ + 1];
        static const char hex[] = "0123456789abcdef";
```

[1]Pseudorandom number generator

```
        char *buffer = hexbuffer[3 & ++bufno], *buf = buffer;
        int i;

        for (i = 0; i < GIT_SHA1_RAWSZ; i++) {
                unsigned int val = *sha1++;
                *buf++ = hex[val >> 4];
                *buf++ = hex[val & 0xf];
        }
        *buf = '\0';

        return buffer;
}
```

(https://github.com/git/git/blob/aa1c6fdf478c023180e5ca5f1658b00a72592dc6/hex.c)

This function returns a pointer to the string containing hexadecimal representation of SHA1 digest (like "4e1243bd22c66e76c2ba9eddc1f91394e57f9f83"). But this is plain C and you can calculate SHA1 for some block, get pointer to the string, then calculate SHA1 for another block, get pointer to the string, and both pointers are still points to the same string buffer containing the result of the second calculation. As a solution, it's possible to allocate/deallocate string buffer each time, but more hackish way is to have several buffers (4 are here) and fill the next each time. The *bufno* variable here is a buffer counter in 0..3 range. Its value increments each time, and its value is also always kept in limits by AND operation (*3 & ++bufno*).

The author of this piece of code (seemingly Linus Torvalds himself) went even further and forgot (?) to initialize *bufno* counter variable, which will have random garbage at the function start. Indeed: no matter, which buffer we are starting each time! This can be mistake which isn't affect correctness of the code, or maybe this is left so intentionally – I don't know.

Perhaps, many of us had used a code like that:

```
cnt=0;

for (...)
{
    do_something();
    cnt=cnt+1;
    if ((cnt % 1000)==0);
        print ("heartbeat: %d processed", cnt);
```

This is how you can use remainder from division for occasional logging: printing will happen once in 1000 steps.

However, another way is:

```
cnt=0

for (...)
{
    do_something();
    cnt=cnt+1;
    if ((cnt&0xfff)==0);
        print ("heartbeat: %d processed", cnt);
```

Now printing will occur once in 0x1000 or 4096 steps. This is also getting remainder from division.

7.3 Modulo inverse, part I

Example: this piece of code divides by 17:

```
#include <stdio.h>
#include <stdint.h>
```

```
uint32_t div17 (uint32_t a)
{
        return a*0xf0f0f0f1;
};

int main()
{
        printf ("%d\n", div17(1700));   // result=100
        printf ("%d\n", div17(34));     // result=2
        printf ("%d\n", div17(2091));   // result=123
};
```

How it works?

Let's imagine, we work on 4-bit CPU, it has 4-bit registers, each can hold a value in 0..15 range.

Now we want to divide by 3 using multiplication. Let's find modulo inverse of 3 using Wolfram Mathematica:

```
In[]:= PowerMod[3, -1, 16]
Out[]= 11
```

This is in fact solution of a $3m = 16k + 1$ equation ($16 = 2^4$):

```
In[]:= FindInstance[3 m == 16 k + 1, {m, k}, Integers]
Out[]= {{m -> 11, k -> 2}}
```

The "magic number" for division by 3 is 11. Multiply by 11 instead of dividing by 3 and you'll get a result (quotient).

This works, let's divide 6 by 3. We can now do this by multiplying 6 by 11, this is 66=0x42, but on 4-bit register, only 0x2 will be left in register ($0x42 \equiv 2 \mod 2^4$). Yes, 2 is correct answer, 6/3=2.

Let's divide 3, 6 and 9 by 3, by multiplying by 11 (m).

```
            |123456789abcdef0|12  ...  f0|123456789abcdef0|12  ...  f0|123456789abcdef0|12  ...  f0|123456789abcdef0|
     m=11   |***********     |    ...    |                |    ...    |                |    ...    |                |
3/3  3m=33  |****************|**  ...  **|*               |    ...    |                |    ...    |                |
6/3  6m=66  |****************|**  ...  **|****************|**  ...  **|**              |    ...    |                |
9/3  9m=99  |****************|**  ...  **|****************|**  ...  **|****************|**  ...  **|***             |
```

A "protruding" asterisk(s) ("*") in the last non-empty chunk is what will be left in 4-bit register. This is 1 in case of 33, 2 if 66, 3 if 99.

In fact, this "protrusion" is defined by 1 in the equation we've solved. Let's replace 1 with 2:

```
In[]:= FindInstance[3 m == 16 k + 2, {m, k}, Integers]
Out[]= {{m -> 6, k -> 1}}
```

Now the new "magic number" is 6. Let's divide 3 by 3. 3*6=18=0x12, 2 will be left in 4-bit register. This is incorrect, we have 2 instead of 1. 2 asterisks are "protruding". Let's divide 6 by 3. 6*6=36=0x24, 4 will be left in the register. This is also incorrect, we now have 4 "protruding" asterisks instead of correct 2.

Replace 1 in the equation by 0, and nothing will "protrude".

7.3.1 No remainder?

Now the problem: this only works for dividends in 3x form, i.e., which can be divided by 3 with no remainder. Try to divide 4 by 3, 4*11=44=0x2c, 12 will be left in register, this is incorrect. The correct quotient is 1.

We can also notice that the 4-bit register is "overflown" during multiplication twice as much as in "incorrect" result in low 4 bits.

Here is what we can do: use only high 4 bits and drop low 4 bits. 4*11=0x2c and 2 is high 4 bits. Divide 2 by 2, this is 1.

Let's "divide" 8 by 3. 8*11=88=0x58. 5/2=2, this is correct answer again.

Now this is the formula we can use on our 4-bit CPU to divide numbers by 3: "x*3 » 4 / 2" or "x*3 » 5". This is the same as almost all modern compilers do instead of integer division, but they do this for 32-bit and 64-bit registers.

7.4 Modulo inverse, part II

A very simple function:

```
int f(int a)
{
        return a/9;
};
```

If compiled by non-optimizing GCC 4.4.1...

Listing 7.3: Non-optimizing GCC 4.4.1

```
       public f
f      proc near

arg_0  = dword ptr  8

       push    ebp
       mov     ebp, esp
       mov     ecx, [ebp+arg_0]
       mov     edx, 954437177 ; 38E38E39h
       mov     eax, ecx
       imul    edx
       sar     edx, 1
       mov     eax, ecx
       sar     eax, 1Fh
       mov     ecx, edx
       sub     ecx, eax
       mov     eax, ecx
       pop     ebp
       retn
f      endp
```

And it can be rewritten back to pure C like that:

```
#include <stdint.h>

uint32_t divide_by_9 (uint32_t a)
{
        return ((uint64_t)a * (uint64_t)954437177) >> 33; // 954437177 = 0x38e38e39
};
```

This function still works, without division operation. How?

From school-level mathematics, we can remember that division by 9 can be replaced by multiplication by $\frac{1}{9}$. In fact, sometimes compilers do so for floating-point arithmetics, for example, `FDIV` instruction in x86 code can be replaced by `FMUL`. At least MSVC 6.0 will replace division by 9 by multiplication by 0.111111... and sometimes it's hard to be sure, what operation was in the original source code.

But when we operate over integer values and integer CPU registers, we can't use fractions. However, we can rework fraction like that:

$$result = \frac{x}{9} = x \cdot \frac{1}{9} = x \cdot \frac{1 \cdot MagicNumber}{9 \cdot MagicNumber}$$

Given the fact that division by 2^n is very fast (using shifts), we now should find that $MagicNumber$, for which the following equation will be true: $2^n = 9 \cdot MagicNumber$.

Division by 2^{32} is somewhat hidden: lower 32-bit of product in EAX is not used (dropped), only higher 32-bit of product (in EDX) is used and then shifted by additional 1 bit.

In other words, the assembly code we have just seen multiplicates by $\frac{954437177}{2^{32+1}}$, or divides by $\frac{2^{32+1}}{954437177}$. To find a divisor we just have to divide numerator by denominator. Using Wolfram Alpha, we can get 8.99999999.... as result (which is close to 9).

Many people miss "hidden" division by 2^{32} or 2^{64}, when lower 32-bit part (or 64-bit part) of product is not used. This is why division by multiplication is difficult to understand at the beginning.

7.5 Reversible linear congruential generator

LCG[2] PRNG is very simple: just multiply seed by some value, add another one and here is a new random number. Here is how it is implemented in MSVC (the source code is not original one and is reconstructed by me):

```
uint32_t state;

uint32_t rand()
{
        state=state*214013+2531011;
        return (state>>16)&0x7FFF;
};
```

The last bit shift is attempt to compensate LCG weakness and we may ignore it so far. Will it be possible to make an inverse function to rand(), which can reverse state back? First, let's try to think, what would make this possible? Well, if state internal variable would be some kind of BigInt or BigNum container which can store infinitely big numbers, then, although state is increasing rapidly, it would be possible to reverse the process. But *state* isn't BigInt/BigNum, it's 32-bit variable, and summing operation is easily reversible on it (just subtract 2531011 at each step). As we may know now, multiplication is also reversible: just multiply the state by modular multiplicative inverse of 214013!

```
#include <stdio.h>
#include <stdint.h>

uint32_t state;

void next_state()
{
        state=state*214013+2531011;
};

void prev_state()
{
        state=state-2531011; // reverse summing operation
        state=state*3115528533; // reverse multiply operation. 3115528533 is modular inverse
            of 214013 in 2^32.
};

int main()[style=customc]
{
        state=12345;

        printf ("state=%d\n", state);
        next_state();
        printf ("state=%d\n", state);
        next_state();
        printf ("state=%d\n", state);
        next_state();
        printf ("state=%d\n", state);

        prev_state();
        printf ("state=%d\n", state);
        prev_state();
        printf ("state=%d\n", state);
        prev_state();
        printf ("state=%d\n", state);
};
```

[2]Linear congruential generator

Wow, that works!

```
state=12345
state=-1650445800
state=1255958651
state=-456978094
state=1255958651
state=-1650445800
state=12345
```

It's hard to find a real-world application of reversible LCG, but this could be the one: a media player with forward/backward buttons. Once shuffle is clicked, random number is generated (number of item to be played). User clicks forward: get a new random number by calculating the next state. User clicks backward: get it by calculating the previous state. Thus, a user could navigate through some "virtual" non-existent (but consistent) playlist, which is even not present in media player's memory!

Or maybe you could navigate a huge labyrinth, so huge that you can't record your path in full. But you can pick some seed for LCG and take left/right turns according to PRNG results. You could go back by reversing LCG's state.

7.6 Getting magic number using extended Euclidean algorithm

Extended Euclidean algorithm can find x/y for given a/b in the diophantine equation: $ax + by = gcd(a, b)$.

x/y are also known as "Bézout coefficients".

However, since a/b are coprime to each other, $gcd(a, b) = 1$, and the algorithm will find x/y for the $ax + by = 1$ equation.

Let's plug $a = 3$ and $b = 2^{16}$ (like if we finding magic constant for 16-bit CPU):

```
#include <assert.h>
#include <stdio.h>
#include <stdint.h>

// copypasted and reworked from https://en.wikipedia.org/wiki/Extended_Euclidean_algorithm
void EGCD (int a, int b)
{
        int s=0; int old_s=1;
        int t=1; int old_t=0;
        int r=b; int old_r=a;
        int tmp;

        while (r!=0)
        {
                int quotient=old_r/r;
                tmp=r; r=old_r - quotient*r; old_r=tmp;
                tmp=s; s=old_s - quotient*s; old_s=tmp;
                tmp=t; t=old_t - quotient*t; old_t=tmp;
        };
        printf ("GCD: %d\n", old_r);
        if (old_r!=1)
        {
                printf("%d and %d are not coprimes!\n", a, b);
                return;
        };
        printf ("Bézout coefficients: %d %d\n", old_s, old_t);
        printf ("quotients by the GCD (s,t): %d %d\n", s, t);

        // see also: https://math.stackexchange.com/q/1310415
        if (old_s>=0 && old_t<=0)
                printf ("corrected coefficients: %d(0x%x) %d(0x%x)\n", old_s, old_s,
                    -old_t, -old_t);
        else if (old_s<=0 && old_t >=0)
```

```
                printf ("corrected coefficients: %d(0x%x) %d(0x%x)\n", old_s+b, old_s
                   +b, -old_t+a, -old_t+a);
        else
                assert(0);
};

void main()
{
        EGCD(3, 0x10000);
};
```

```
GCD: 1
Bézout coefficients: -21845 1
quotients by the GCD (s,t): 65536 -3
corrected coefficients: 43691(0xaaab) 2(0x2)
```

That means, x=-21845, y=1. Is it correct? $-21845 * 3 + 65536 * 1 = 1$

65536-21845=0xaaab, and this is the magic number for division by 3 on 16-bit CPU.

$GCD(any\_odd\_number, 2^n) = 1$ for any values. Hence, we can find a magic number for any even number. But, this is not true for even numbers. We can't find magic coefficient for even number like 10. But you can find it for 5, and then add additional division by 2 using shift right.

If you try to compile $\frac{x}{10}$ code, GCC can do it like:

```
push   %ebp
mov    %esp,%ebp
mov    0x8(%ebp),%eax
mov    $0xcccccccd,%edx
mul    %edx
mov    %edx,%eax
shr    $0x3,%eax
pop    %ebp
ret
```

This is in fact the magic number for division by 5. And there is 3 instead of 2 in the SHR instruction, so the result is divided by 2.

Extended Euclidean algorithm is probably an efficient way of finding magic number, but needless to say, this equation can be solved using other ways. In "SAT/SMT by Example"[3]you can find a method of finding "magic number" using SMT-solver.

[3]https://sat-smt.codes/

Chapter 8

Probability

8.1 Text strings right in the middle of compressed data

You can download Linux kernels and find English words right in the middle of compressed data:

```
% wget https://www.kernel.org/pub/linux/kernel/v4.x/linux-4.10.2.tar.gz

% xxd -g 1 -seek 0x515c550 -l 0x30 linux-4.10.2.tar.gz

0515c550: c5 59 43 cf 41 27 85 54 35 4a 57 90 73 89 b7 6a  .YC.A'.T5JW.s..j
0515c560: 15 af 03 db 20 df 6a 51 f9 56 49 52 55 53 3d da  .... .jQ.VIRUS=.
0515c570: 0e b9 29 24 cc 6a 38 e2 78 66 09 33 72 aa 88 df  ..)$.j8.xf.3r...
```

```
% wget https://cdn.kernel.org/pub/linux/kernel/v2.3/linux-2.3.3.tar.bz2

% xxd -g 1 -seek 0xa93086 -l 0x30 linux-2.3.3.tar.bz2

00a93086: 4d 45 54 41 4c cd 44 45 2d 2c 41 41 54 94 8b a1  METAL.DE-,AAT...
00a93096: 5d 2b d8 d0 bd d8 06 91 74 ab 41 a0 0a 8a 94 68  ]+......t.A....h
00a930a6: 66 56 86 81 68 0d 0e 25 6b b6 80 a4 28 1a 00 a4  fV..h..%k...(...
```

One of Linux kernel patches in compressed form has the “Linux” word itself (not a bad self-referential joke):

```
% wget https://cdn.kernel.org/pub/linux/kernel/v4.x/testing/patch-4.6-rc4.gz

% xxd -g 1 -seek 0x4d03f -l 0x30 patch-4.6-rc4.gz

0004d03f: c7 40 24 bd ae ef ee 03 2c 95 dc 65 eb 31 d3 f1  .@$.....,..e.1..
0004d04f: 4c 69 6e 75 78 f2 f3 70 3c 3a bd 3e bd f8 59 7e  Linux..p<:.>..Y~
0004d05f: cd 76 55 74 2b cb d5 af 7a 35 56 d7 5e 07 5a 67  .vUt+...z5V.^.Zg
```

Other English words I’ve found in other compressed Linux kernel trees:

```
linux-4.6.2.tar.gz: [maybe] at 0x68e78ec
linux-4.10.14.tar.xz: [OCEAN] at 0x6bf0a8
linux-4.7.8.tar.gz: [FUNNY] at 0x29e6e20
linux-4.6.4.tar.gz: [DRINK] at 0x68dc314
linux-2.6.11.8.tar.bz2: [LUCKY] at 0x1ab5be7
linux-3.0.68.tar.gz: [BOOST] at 0x11238c7
linux-3.0.16.tar.bz2: [APPLE] at 0x34c091
linux-3.0.26.tar.xz: [magic] at 0x296f7d9
linux-3.11.8.tar.bz2: [TRUTH] at 0xf635ba
linux-3.10.11.tar.bz2: [logic] at 0x4a7f794
```

There is a nice illustration of apophenia and pareidolia (human's mind ability to see faces in clouds, etc) in Lurkmore, Russian counterpart of Encyclopedia Dramatica. As they wrote in the article about electronic voice phenomenon[1], you can open any long enough compressed file in hex editor and find well-known 3-letter Russian obscene word, and you'll find it a lot: but that means nothing, just a mere coincidence.

And I was interested in calculation, how big compressed file must be to contain all possible 3-letter, 4-letter, etc, words? In my naive calculations, I've got this: probability of the first specific byte in the middle of compressed data stream with maximal entropy is $\frac{1}{256}$, probability of the 2nd is also $\frac{1}{256}$, and probability of specific byte pair is $\frac{1}{256 \cdot 256} = \frac{1}{256^2}$. Probability of specific triple is $\frac{1}{256^3}$. If the file has maximal entropy (which is almost unachievable, but ...) and we live in an ideal world, you've got to have a file of size just $256^3 = 16777216$, which is 16-17MB. You can check: get any compressed file, and use *rafind2* to search for any 3-letter word (not just that Russian obscene one).

It took $\approx$ 8-9 GB of my downloaded movies/TV series files to find the word "beer" in them (case sensitive). Perhaps, these movies wasn't compressed good enough? This is also true for a well-known 4-letter English obscene word.

My approach is naive, so I googled for mathematically grounded one, and have find this question: "Time until a consecutive sequence of ones in a random bit sequence" [2]. The answer is: $(p^{-n}-1)/(1-p)$, where p is probability of each event and n is number of consecutive events. Plug $\frac{1}{256}$ and 3 and you'll get almost the same as my naive calculations.

So any 3-letter word can be found in the compressed file (with ideal entropy) of length $256^3 = \approx 17MB$, any 4-letter word — $256^4 = 4.7GB$ (size of DVD). Any 5-letter word — $256^5 = \approx 1TB$.

For the piece of text you are reading now, I mirrored the whole kernel.org website (hopefully, sysadmins can forgive me), and it has $\approx$ 430GB of compressed Linux Kernel source trees. It has enough compressed data to contain these words, however, I cheated a bit: I searched for both lowercase and uppercase strings, thus compressed data set I need is almost halved.

This is quite interesting thing to think about: 1TB of compressed data with maximal entropy has all possible 5-byte chains, but the data is encoded not in chains itself, but in the order of chains (no matter of compression algorithm, etc).

Now the information for gamblers: one should throw a dice $\approx$ 42 times to get a pair of six, but no one will tell you, when exactly this will happen. I don't remember, how many times coin was tossed in the "Rosencrantz & Guildenstern Are Dead" movie, but one should toss it $\approx$ 2048 times and at some point, you'll get 10 heads in a row, and at some other point, 10 tails in a row. Again, no one will tell you, when exactly this will happen.

Compressed data can also be treated as a stream of random data, so we can use the same mathematics to determine probabilities, etc.

If you can live with strings of mixed case, like "bEeR", probabilities and compressed data sets are much lower: $128^3 = 2MB$ for all 3-letter words of mixed case, $128^4 = 268MB$ for all 4-letter words, $128^5 = 34GB$ for all 5-letter words, etc.

Moral of the story: whenever you search for some patterns, you can find it in the middle of compressed blob, but that means nothing else then coincidence. In philosophical sense, this is a case of selection/confirmation bias: you find what you search for in "The Library of Babel" [3].

Also, a citation from a Marten Gardner's book about randomness of π:

> So far pi has passed all statistical tests for randomness. This is disconcerting to those who feel that a curve so simple and beautiful as the circle should have a less-disheveled ratio between the way around and the way across, but most mathematicians believe that no pattern or order of any sort will ever be found in pi's decimal expansion. Of course the digits are not random in the sense that they represent pi, but then in this sense neither are the million random digits that have been published by the Rand Corporation of California. They too represent a single number, and an integer at that. If it is true that the digits in pi are random, perhaps we are justified in stating a paradox somewhat similar to the assertion that if a group of monkeys pound long enough on typewriters, they will eventually type all the plays of Shakespeare. Stephen Barr has pointed out that if you set no limit to the accuracy with which two bars can be constructed and measured, then those two bars, without any markings on them, can communicate the entire Encyclopaedia Britannica. One bar is taken as unity. The other differs

[1] `http://archive.is/gYnFL`

[2] `http://math.stackexchange.com/questions/27989/time-until-a-consecutive-sequence-of-ones-in-a-random-bit-sequence/27991#27991`

[3] A short story by Jorge Luis Borges

from unity by a fraction that is expressed as a very long decimal. This decimal codes the Britannica by the simple process of assigning a different number (excluding zero as a digit in the number) to every word and mark of punctuation in the language. Zero is used to separate the code numbers. Obviously the entire Britannica can now be coded as a single, but almost inconceivably long, number. Put a decimal point in front of this number, add 1, and you have the length of the second of Barr's bars. Where does pi come in? Well, if the digits in pi are really random, then somewhere in this infinite pie there should be a slice that contains the Britannica; or, for that matter, any book that has been written, will be written, or could be written.

(Martin Gardner – New Mathematical Diversions[4])

Also:

A sequence of six 9's occurs in the decimal representation of the number pi (π), starting at the 762nd decimal place. It has become famous because of the mathematical coincidence and because of the idea that one could memorize the digits of π up to that point, recite them and end with "nine nine nine nine nine nine and so on", which seems to suggest that π is rational. The earliest known mention of this idea occurs in Douglas Hofstadter's 1985 book Metamagical Themas, where Hofstadter states

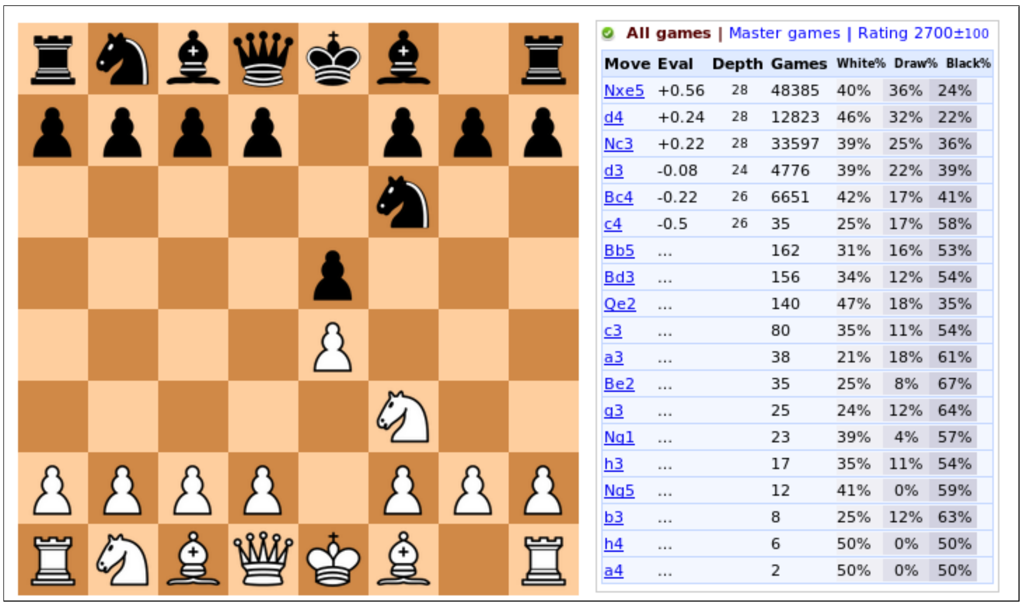
I myself once learned 380 digits of π, when I was a crazy high-school kid. My never-attained ambition was to reach the spot, 762 digits out in the decimal expansion, where it goes "999999", so that I could recite it out loud, come to those six 9's, and then impishly say, "and so on!"
— Douglas Hofstadter, Metamagical Themas

(https://en.wikipedia.org/wiki/Six_nines_in_pi)

8.2 Autocomplete using Markov chains

TL;DR: collect statistics, for a given natural language, what words come most often after a word/pair of words/triplet of words.

What are most chess moves played after 1.e4 e5 2.Nf3 Nf6? A big database of chess games can be queried, showing statistics:


All games | Master games | Rating 2700±100

Move	Eval	Depth	Games	White%	Draw%	Black%
Nxe5	+0.56	28	48385	40%	36%	24%
d4	+0.24	28	12823	46%	32%	22%
Nc3	+0.22	28	33597	39%	25%	36%
d3	-0.08	24	4776	39%	22%	39%
Bc4	-0.22	26	6651	42%	17%	41%
c4	-0.5	26	35	25%	17%	58%
Bb5	...		162	31%	16%	53%
Bd3	...		156	34%	12%	54%
Qe2	...		140	47%	18%	35%
c3	...		80	35%	11%	54%
a3	...		38	21%	18%	61%
Be2	...		35	25%	8%	67%
g3	...		25	24%	12%	64%
Ng1	...		23	39%	4%	57%
h3	...		17	35%	11%	54%
Ng5	...		12	41%	0%	59%
b3	...		8	25%	12%	63%
h4	...		6	50%	0%	50%
a4	...		2	50%	0%	50%

(from https://chess-db.com/)

Statistics shown is just number of games, where a corresponding 3rd move was played.

[4] https://books.google.com/books?id=VEOFEAAAQBAJ

The same database can be made for natural language words.

8.2.1 Dissociated press

This is a well-known joke: https://en.wikipedia.org/wiki/Dissociated_press, https://en.wikipedia.org/wiki/SCIgen, https://en.wikipedia.org/wiki/Mark_V._Shaney.

I wrote a Python script, took Sherlock Holmes stories from Gutenberg library...

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-

# python 3! due to random.choices()

import operator, random
from collections import defaultdict

with open ("all.txt", "r") as myfile:
    data=myfile.read()

sentences=data.lower().replace('\r',' ').replace('\n',' ').replace('?','.').replace('
   !','.').replace('“','.').replace('”','.').replace("\"",".").replace('‘',' ').
   replace('-',' ').replace('’',' ').replace('\'',' ').split(".")

def remove_empty_words(l):
    return list(filter(lambda a: a != '', l))

# key=list of words, as a string, delimited by space
# (I would use list of strings here as key, but list in not hashable)
# val=dict, k: next word; v: occurrences
first={}
second={}

def update_occ(d, seq, w):
    if seq not in d:
        d[seq]=defaultdict(int)

    d[seq][w]=d[seq][w]+1

for s in sentences:
    words=s.replace(',',' ').split(" ")
    words=remove_empty_words(words)
    if len(words)==0:
        continue
    for i in range(len(words)):
        if i>=1:
            update_occ(first, words[i-1], words[i])
        if i>=2:
            update_occ(second, words[i-2]+" "+words[i-1], words[i])

"""
print ("first table:")
for k in first:
    print (k)
    # https://stackoverflow.com/questions/613183/how-do-i-sort-a-dictionary-by-value
    s=sorted(first[k].items(), key=operator.itemgetter(1), reverse=True)
    print (s[:20])
    print ("")
"""
"""
print ("second table:")
for k in second:
    print (k)
    # https://stackoverflow.com/questions/613183/how-do-i-sort-a-dictionary-by-value
    s=sorted(second[k].items(), key=operator.itemgetter(1), reverse=True)
    print (s[:20])
    print ("")
"""

text=["it", "is"]

# https://docs.python.org/3/library/random.html#random.choice
def gen_random_from_tbl(t):
    return random.choices(list(t.keys()), weights=list(t.values()))[0]

text_len=len(text)

# generate at most 100 words:
for i in range(200):

    last_idx=text_len-1

    tmp=text[last_idx-1]+" "+text[last_idx]
    if tmp in second:
        new_word=gen_random_from_tbl(second[tmp])
    else:
        # fall-back to 1st order
        tmp2=text[last_idx]
        if tmp2 not in first:
            # dead-end
            break
        new_word=gen_random_from_tbl(first[tmp2])

    text.append(new_word)
    text_len=text_len+1

print (" ".join(text))
```

And here are some 1st-order Markov chains. First part is a first word. Second is a list of words + probabilities of appearance of each one, in the Sherlock Holmes stories. However, probabilities are in form of words' numbers.

In other word, how often each word appears after a word?

```
return
[('to', 28), ('or', 12), ('of', 8), ('the', 8), ('for', 5), ('with', 4), ('by', 4),
   ('journey', 4), ('and', 3), ('when', 2), ('ticket', 2), ('from', 2), ('at', 2), ('
   you', 2), ('i', 2), ('since', 1), ('on', 1), ('caused', 1), ('but', 1), ('it', 1)]

of
[('the', 3206), ('a', 680), ('his', 584), ('this', 338), ('it', 304), ('my', 295), ('
   course', 225), ('them', 167), ('that', 138), ('your', 137), ('our', 135), ('us',
   122), ('her', 116), ('him', 111), ('an', 105), ('any', 102), ('these', 92), ('
   which', 89), ('all', 82), ('those', 75)]

by
[('the', 532), ('a', 164), ('his', 67), ('my', 39), ('no', 34), ('this', 31), ('an',
   30), ('which', 29), ('your', 21), ('that', 19), ('one', 17), ('all', 17), ('jove',
    16), ('some', 16), ('sir', 13), ('its', 13), ('him', 13), ('their', 13), ('it',
   11), ('her', 10)]

this
[('is', 177), ('agreement', 96), ('morning', 81), ('was', 61), ('work', 60), ('man',
   56), ('case', 43), ('time', 39), ('matter', 37), ('ebook', 36), ('way', 36), ('
   fellow', 28), ('room', 27), ('letter', 24), ('one', 24), ('i', 23), ('young', 21),
    ('very', 20), ('project', 18), ('electronic', 18)]

no
[('doubt', 179), ('one', 134), ('sir', 61), ('more', 56), ('i', 55), ('no', 42), ('
   other', 36), ('means', 36), ('sign', 34), ('harm', 23), ('reason', 22), ('
   difficulty', 22), ('use', 21), ('idea', 20), ('longer', 20), ('signs', 18), ('good
   ', 17), ('great', 16), ('trace', 15), ('man', 15)]
```

Now some snippets from 2nd-order Markov chains.

```
the return
[('of', 8), ('track', 1), ('to', 1)]

return of
[('sherlock', 6), ('lady', 1), ('the', 1)]

for the
[('use', 22), ('first', 12), ('purpose', 9), ('sake', 8), ('rest', 7), ('best', 7),
   ('crime', 6), ('evening', 6), ('ebooks', 6), ('limited', 6), ('moment', 6), ('day
   ', 5), ('future', 5), ('last', 5), ('world', 5), ('time', 5), ('loss', 4), ('
   second', 4), ('night', 4), ('remainder', 4)]

use of
[('the', 15), ('anyone', 12), ('it', 6), ('project', 6), ('and', 6), ('having', 2),
   ('this', 1), ('my', 1), ('a', 1), ('horses', 1), ('some', 1), ('arguing', 1), ('
   troubling', 1), ('artificial', 1), ('invalids', 1), ('cocaine', 1), ('disguises',
   1), ('an', 1), ('our', 1)]

you may
[('have', 17), ('copy', 13), ('be', 13), ('do', 9), ('choose', 7), ('use', 6), ('
   obtain', 6), ('convert', 6), ('charge', 6), ('demand', 6), ('remember', 5), ('find
   ', 5), ('take', 4), ('think', 3), ('possibly', 3), ('well', 3), ('know', 3), ('not
   ', 3), ('say', 3), ('imagine', 3)]

the three
[('of', 8), ('men', 4), ('students', 2), ('glasses', 2), ('which', 1), ('strips', 1),
    ('is', 1), ('he', 1), ('gentlemen', 1), ('enterprising', 1), ('massive', 1), ('
   quarter', 1), ('randalls', 1), ('fugitives', 1), ('animals', 1), ('shrieked', 1),
   ('other', 1), ('murderers', 1), ('fir', 1), ('at', 1)]

it was
[('a', 179), ('the', 78), ('not', 68), ('only', 40), ('in', 30), ('evident', 28), ('
   that', 28), ('all', 25), ('to', 21), ('an', 18), ('my', 17), ('at', 17), ('
   impossible', 17), ('indeed', 15), ('no', 15), ('quite', 15), ('he', 14), ('of',
   14), ('one', 12), ('on', 12)]
```

Now two words is a key in dictionary. And we see here an answer for the question ”how often each words appears after a sequence of two words?”

Now let’s generate some rubbish:

```
it is just to the face of granite still cutting the lower part of the shame which i
   would draw up so as i can t have any visitors if he is passionately fond of a dull
    wrack was drifting slowly in our skin before long vanished in front of him of
   this arch rebel lines the path which i had never set foot in the same with the
   heads of these have been supplemented or more waiting to tell me all that my
   assistant hardly knowing whether i had 4 pounds a month however is foreign to the
   other hand were most admirable but because i know why he broke into a curve by the
    crew had thought it might have been on a different type they were about to set
   like a plucked chicken s making the case which may help to play it at the door and
    called for the convenience of a boat sir maybe i could not be made to draw some
   just inference that a woman exactly corresponding to the question was how a miners
```

```
     camp had been dashed savagely against the rock in front of the will was drawn
    across the golden rays and it

it is the letter was composed of a tall stout official had come the other way that
    she had been made in this i expect that we had news that the office of the mud
    settling or the most minute exactness and astuteness represented as i come to see
    the advantages which london then and would i could not speak before the public
    domain ebooks in compliance with any particular paper edition as to those letters
    come so horribly to his feet upon the wet clayey soil but since your wife and of
    such damage or cannot be long before the rounds come again whenever she might be
    useful to him so now my fine fellow will trouble us again and again and making a
    little wizened man darted out of the baskervilles *** produced by roger squires
    updated editions will be the matter and let out forty three diamonds of the attack
     we carried him into the back and white feather were but a terrible fellow he is
    not for your share in an instant holmes clapped his hands and play with me anyhow
    i d ha known you under that name in a considerable treasure was hid for no other

it is the unofficial force the shutter open but so far as to what i say to me like
    that which is rather too far from at ease for i knew that we were driving; but
    soon it came just as he opened it myself i was not a usual signal between you and
    you thought it might be removed to a magistrate than upon the luminous screen of
    the one word would he have wanted there are business relations between him and we
    listened to his eyes to the spot as soon as their master s affairs of life since
    she was but one day we hoped would screen me from under his arm chair of shining
    red leather chair his flask in his chair and gave me his name is sherlock holmes
    took each face of a barmaid in bristol and marry her but learned from dr

it is selden the criminal or lunatic who had left a match might mar my career had
    reached the tower of the crime was committed before twelve foolish tradesmen in a
    bad lot though the authorities are excellent at amassing facts though what its
    object might be a better nerve for the creature flapped and struggled and writhed
    his hands in her bosom lady hilda was down on the table and the curious will so
    suddenly upon the floor were now nearly five thousand pounds will be impossible
    but i struck him down and roared into the shadow of the thing seemed simplicity
    itself said i you seem most fortunate in having every characteristic of the place
    is deserted up to keep some clandestine appointment and found as i have no desire
    to settle this little matter of fact of absolute ignorance and he with a similar
    pearl without any light upon what he had found ourselves at the time and my heart
    sank for barrymore at the close of november and holmes fears came to think over
    the afghan campaign yet shown by this time at which he now and i have seen his
    death this morning he walked straight into
```

By first look, these pieces of text are visually OK, but it is senseless. Some people (including me) find it amusing.

Spammers also use this technique to make email message visually similar to a meaningful text, albeit it is not meaningful at all, rather absurdic and funny.

8.2.2 Autocomplete

It’s surprising how easy this can be turned into something rather practically useful.

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-

import operator
from collections import defaultdict

with open ("all.txt", "r") as myfile:
    data=myfile.read()

sentences=data.lower().replace('\r',' ').replace('\n',' ').replace('?','.').replace('
    !','.').replace('“','.').replace('”','.').replace("\"",".").replace('‘',' ').
    replace('-',' ').replace('’',' ').replace('\'',' ').split(".")

def remove_empty_words(l):
    return list(filter(lambda a: a != '', l))

# key=list of words, as a string, delimited by space
# (I would use list of strings here as key, but list in not hashable)
# val=dict, k: next word; v: occurrences
first={}
second={}
third={}

def update_occ(d, seq, w):
    if seq not in d:
        d[seq]=defaultdict(int)

    d[seq][w]=d[seq][w]+1

for s in sentences:
    words=s.replace(',',' ').split(" ")
    words=remove_empty_words(words)
    if len(words)==0:
        continue
    for i in range(len(words)):
        # only two words available:
        if i>=1:
            update_occ(first, words[i-1], words[i])
        # three words available:
        if i>=2:
            update_occ(second, words[i-2]+" "+words[i-1], words[i])
        # four words available:
        if i>=3:
            update_occ(third, words[i-3]+" "+words[i-2]+" "+words[i-1], words[i])

"""
print ("third table:")
for k in third:
    print (k)
    # https://stackoverflow.com/questions/613183/how-do-i-sort-a-dictionary-by-value
    s=sorted(third[k].items(), key=operator.itemgetter(1), reverse=True)
    print (s[:20])
    print ("")
"""

test="i can tell"
#test="who did this"
#test="she was a"
#test="he was a"
#test="i did not"
#test="all that she"
#test="have not been"
#test="who wanted to"
#test="he wanted to"
#test="wanted to do"
#test="it is just"
#test="you will find"
#test="you shall"
#test="proved to be"

test_words=test.split(" ")
```

```
test_len=len(test_words)
last_idx=test_len-1

def print_stat(t):
    total=float(sum(t.values()))

    # https://stackoverflow.com/questions/613183/how-do-i-sort-a-dictionary-by-value
    s=sorted(t.items(), key=operator.itemgetter(1), reverse=True)
    # take 5 from each sorted table
    for pair in s[:5]:
        print ("%s %d%%" % (pair[0], (float(pair[1])/total)*100))

if test_len>=3:
    tmp=test_words[last_idx-2]+" "+test_words[last_idx-1]+" "+test_words[last_idx]
    if tmp in third:
        print ("* third order. for sequence:",tmp)
        print_stat(third[tmp])

if test_len>=2:
    tmp=test_words[last_idx-1]+" "+test_words[last_idx]
    if tmp in second:
        print ("* second order. for sequence:", tmp)
        print_stat(second[tmp])

if test_len>=1:
    tmp=test_words[last_idx]
    if tmp in first:
        print ("* first order. for word:", tmp)
        print_stat(first[tmp])
print ("")
```

First, let's also make 3rd-order Markov chains tables. There are some snippets from it:

```
the return of
[('sherlock', 6), ('lady', 1), ('the', 1)]

the use of
[('the', 13), ('anyone', 12), ('project', 6), ('having', 2), ('a', 1), ('horses', 1),
    ('some', 1), ('troubling', 1), ('artificial', 1), ('invalids', 1), ('disguises',
   1), ('it', 1)]

of the second
[('stain', 5), ('floor', 3), ('page', 1), ('there', 1), ('person', 1), ('party', 1),
   ('day', 1)]

it was in
[('the', 9), ('vain', 5), ('a', 4), ('his', 2), ('83', 1), ('one', 1), ('motion', 1),
    ('truth', 1), ('my', 1), ('this', 1), ('march', 1), ('january', 1), ('june', 1),
   ('me', 1)]

was in the
[('room', 3), ('habit', 3), ('house', 2), ('last', 2), ('loft', 2), ('spring', 1), ('
   train', 1), ('bar', 1), ('power', 1), ('immediate', 1), ('year', 1), ('midst', 1),
    ('forenoon', 1), ('centre', 1), ('papers', 1), ('best', 1), ('darkest', 1), ('
   prime', 1), ('hospital', 1), ('nursery', 1)]

murder of the
[('honourable', 1), ('discoverer', 1)]

particulars of the
[('crime', 1), ('inquest', 1), ('habits', 1), ('voyage', 1)]
```

```
the death of
[('the', 8), ('sir', 8), ('captain', 3), ('their', 2), ('this', 2), ('his', 2), ('her
    ', 2), ('dr', 2), ('sherlock', 1), ('mrs', 1), ('a', 1), ('van', 1), ('that', 1),
    ('drebber', 1), ('mr', 1), ('stangerson', 1), ('two', 1), ('selden', 1), ('one',
    1), ('wallenstein', 1)]

one of the
[('most', 23), ('best', 3), ('main', 3), ('oldest', 2), ('greatest', 2), ('numerous',
     2), ('corners', 2), ('largest', 2), ('papers', 2), ('richest', 2), ('servants',
    2), ('moor', 2), ('finest', 2), ('upper', 2), ('very', 2), ('broad', 2), ('side',
    2), ('highest', 2), ('australian', 1), ('great', 1)]

so far as
[('i', 17), ('to', 8), ('that', 2), ('the', 2), ('it', 2), ('was', 1), ('we', 1), ('
    they', 1), ('he', 1), ('you', 1), ('a', 1), ('your', 1), ('this', 1), ('his', 1),
    ('she', 1)]
```

You see, they looks as more precise, but tables are just smaller. You can't use them to generate rubbish. 1st-order tables big, but ”less precise”.

And here I test some 3-words queries, like as if they inputted by user:

```
"i can tell"

* third order. for sequence: i can tell
you 66%
my 16%
them 16%
* second order. for sequence: can tell
you 23%
us 17%
me 11%
your 11%
this 5%
* first order. for word: tell
you 35%
me 25%
us 6%
him 5%
the 4%

"she was a"

* third order. for sequence: she was a
blonde 12%
child 6%
passenger 6%
free 6%
very 6%
* second order. for sequence: was a
very 4%
man 3%
small 3%
long 2%
little 2%
* first order. for word: a
very 2%
man 2%
little 2%
few 2%
small 1%
```

```
"he was a"

* third order. for sequence: he was a
man 11%
very 7%
young 3%
tall 3%
middle 2%
* second order. for sequence: was a
very 4%
man 3%
small 3%
long 2%
little 2%
* first order. for word: a
very 2%
man 2%
little 2%
few 2%
small 1%

"i did not"

* third order. for sequence: i did not
know 22%
say 9%
even 3%
tell 3%
mind 3%
* second order. for sequence: did not
know 11%
take 3%
wish 3%
go 2%
say 2%
* first order. for word: not
a 4%
be 4%
have 3%
been 3%
to 2%

"all that she"

* third order. for sequence: all that she
said 100%
* second order. for sequence: that she
had 22%
was 19%
would 7%
is 5%
has 5%
* first order. for word: she
was 12%
had 10%
is 5%
said 3%
would 3%

"have not been"
```

```
* third order. for sequence: have not been
able 25%
here 25%
employed 12%
shadowed 12%
personally 12%
* second order. for sequence: not been
for 9%
in 6%
there 6%
a 4%
slept 4%
* first order. for word: been
a 5%
in 4%
the 2%
so 2%
taken 1%

"who wanted to"

* third order. for sequence: who wanted to
see 100%
* second order. for sequence: wanted to
see 15%
know 15%
speak 10%
ask 10%
hide 5%
* first order. for word: to
the 11%
be 6%
me 4%
his 2%
my 2%

"he wanted to"

* third order. for sequence: he wanted to
know 50%
do 50%
* second order. for sequence: wanted to
see 15%
know 15%
speak 10%
ask 10%
hide 5%
* first order. for word: to
the 11%
be 6%
me 4%
his 2%
my 2%

"wanted to do"

* third order. for sequence: wanted to do
the 100%
* second order. for sequence: to do
with 23%
```

```
so 14%
it 10%
the 4%
and 3%
* first order. for word: do
you 24%
not 20%
with 6%
so 4%
it 4%

"it is just"

* third order. for sequence: it is just
possible 42%
as 33%
killing 4%
such 4%
in 4%
* second order. for sequence: is just
possible 25%
as 22%
the 8%
a 8%
to 5%
* first order. for word: just
as 13%
now 5%
a 4%
to 3%
the 2%

"you will find"

* third order. for sequence: you will find
that 20%
it 20%
me 8%
the 8%
a 6%
* second order. for sequence: will find
it 19%
that 17%
the 9%
me 7%
a 5%
* first order. for word: find
that 13%
the 10%
it 9%
out 8%
a 6%

"you shall"

* second order. for sequence: you shall
have 15%
see 12%
know 12%
hear 9%
not 9%
```

```
* first order. for word: shall
be 20%
have 7%
not 4%
see 3%
i 3%

"proved to be"

* third order. for sequence: proved to be
a 42%
the 14%
too 4%
something 4%
of 4%
* second order. for sequence: to be
a 11%
the 3%
in 3%
able 2%
so 2%
* first order. for word: be
a 7%
the 4%
in 2%
of 2%
no 2%
```

Perhaps, results from all 3 tables can be combined, with the data from 3rd order table used in highest priority (or weight).

And this is it — this can be shown to user. Aside of Conan Doyle works, your software can collect user's input to adapt itself for user's lexicon, slang, memes, etc. Of course, user's "tables" should be used with highest priority.

I have no idea, what Apple/Android devices using for hints generation, when user input text, but this is what I would use as a first idea.

As a bonus, this can be used for language learners, to get the idea, how a word is used in a sentence.

8.2.3 Further work

Comma can be a separator as well, etc...

8.2.4 The files

... including Conan Doyle's stories (2.5M). https://αβγ.ελ/current_tree/prob/markov. Surely, any other texts can be used, in any language...

Another related post is about typos: https://yurichev.com/blog/fuzzy_string/.

8.2.5 Read more

Meaningful Random Headlines by Markov Chain

A discussion at hacker news

This wonderful chart based on lyrics of popular songs reflects the idea perfectly: https://αβγ.ελ/current_tree/prob/markov/khi7r
Found on Reddit: https://www.reddit.com/r/dataisbeautiful/comments/eq6s6j/oc_i_made_this_chart_of_some_songs/, https://archive.is/jgR6I.

8.2.6 Another thought-provoking example

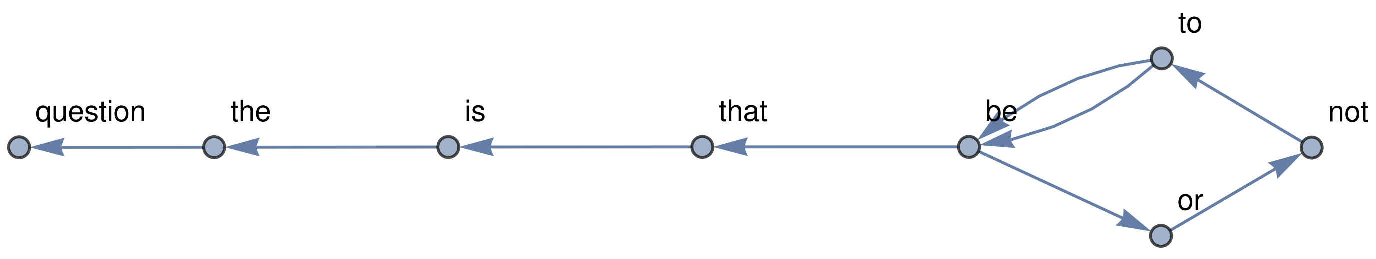
Found in *Wolfram Mathematica Tutorial Collection: Graph Drawing*[5]:

[5]https://library.wolfram.com/infocenter/Books/8508/, https://library.wolfram.com/infocenter/Books/8508/GraphDrawing.pdf.

As they say, "this generates a graph by linking words in a text with subsequent words":

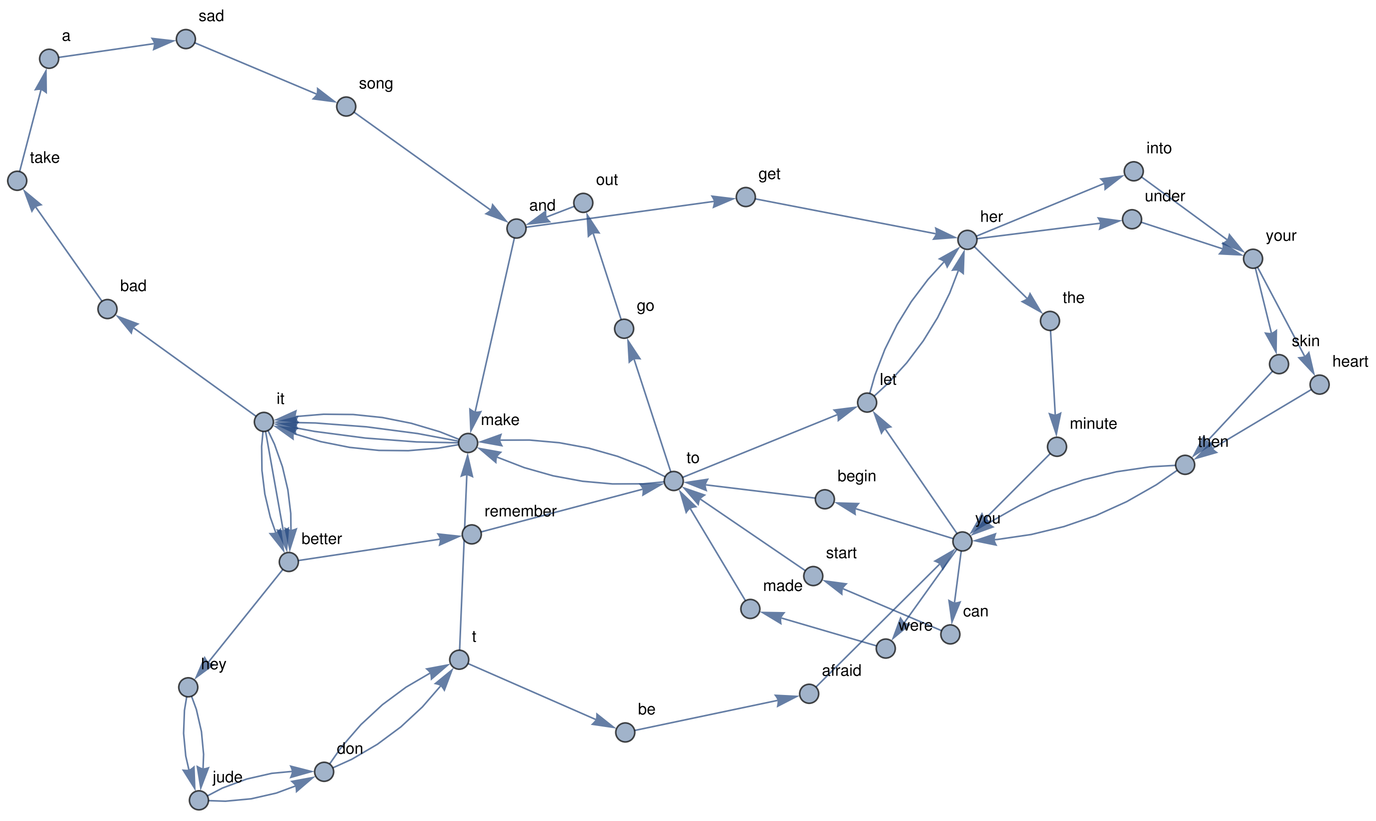
```
TextPlot[w_String] :=
  GraphPlot[(x = Map[ToLowerCase, StringCases[w, WordCharacter ..]];
    g = Thread[Drop[x, -1] -> Drop[x, 1]];
    g), DirectedEdges -> True, VertexLabeling -> True];

TextPlot["to be or not to be, that is the question"]
```

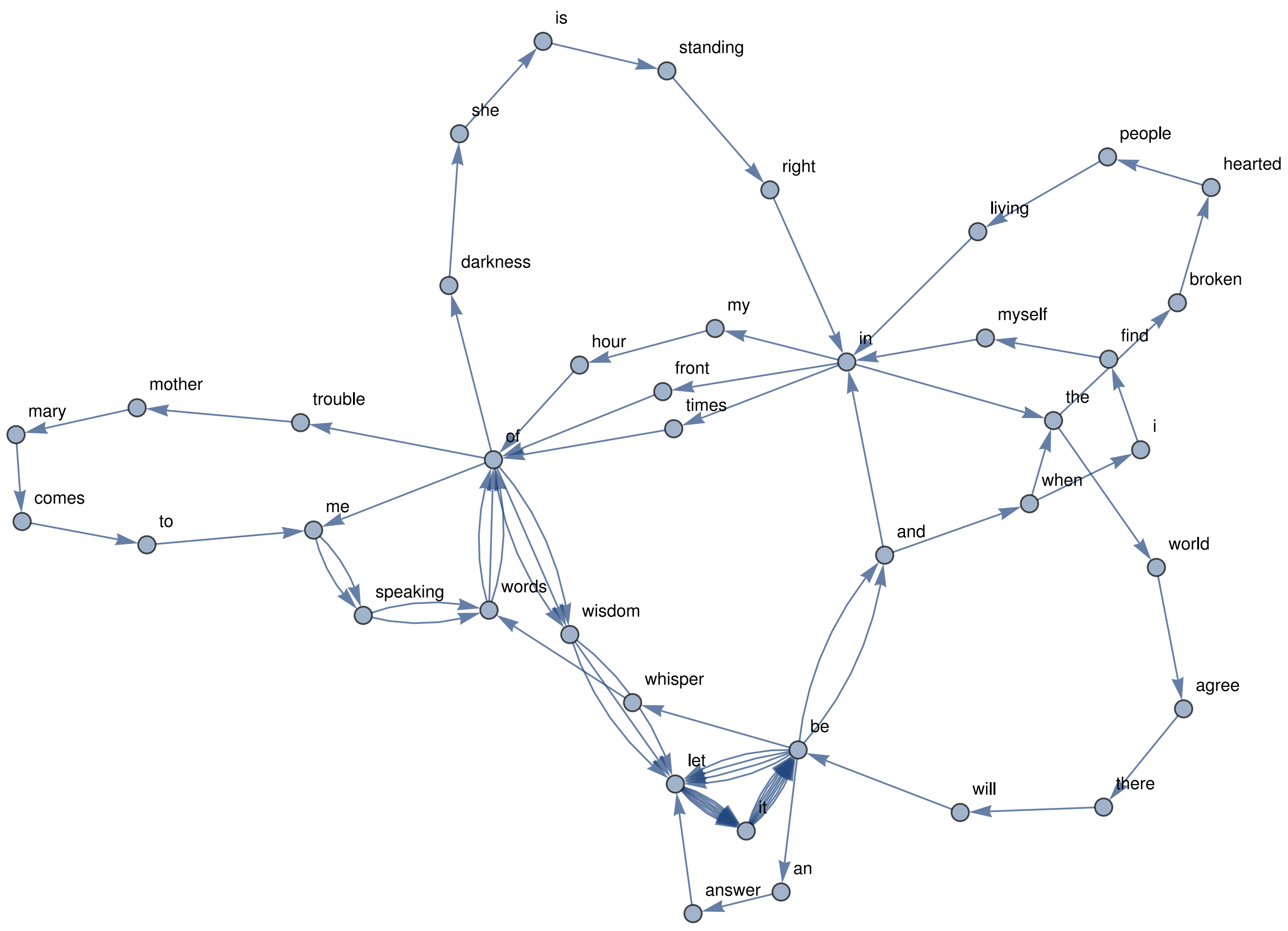


Nice, so I wanted to experiment further, taking some popular songs:

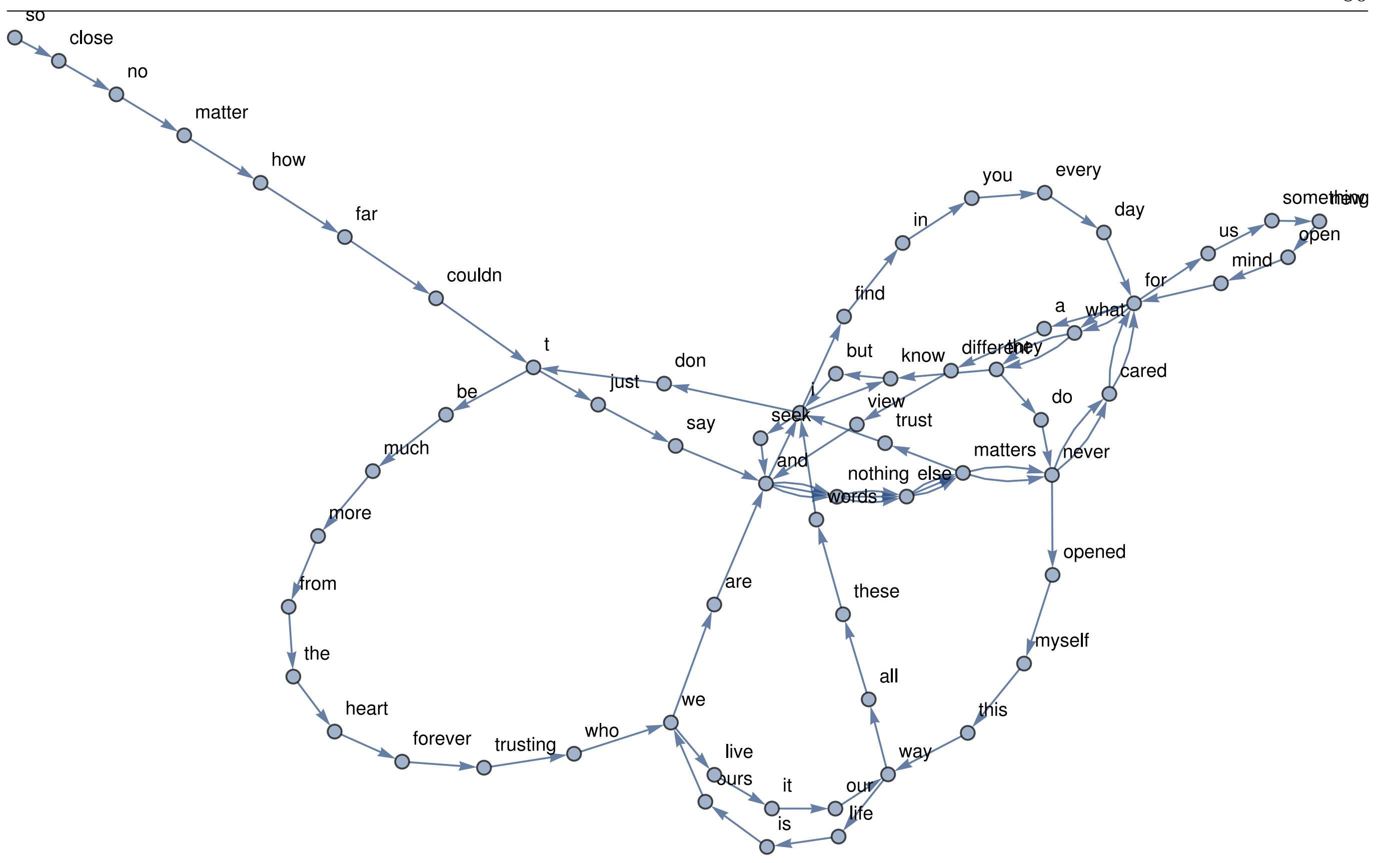
```
TextPlot["Hey Jude, don't make it bad.
 Take a sad song and make it better.
 Remember to let her into your heart,
 Then you can start to make it better.
 Hey Jude, don't be afraid.
 You were made to go out and get her.
 The minute you let her under your skin,
 Then you begin to make it better."]
```



```
TextPlot["When I find myself in times of trouble, Mother Mary comes \
to me
 Speaking words of wisdom, let it be
 And in my hour of darkness she is standing right in front of me
 Speaking words of wisdom, let it be
 Let it be, let it be, let it be, let it be
 Whisper words of wisdom, let it be
 And when the broken hearted people living in the world agree
 There will be an answer, let it be"]
```



```
TextPlot["So close, no matter how far
 Couldn't be much more from the heart
 Forever trusting who we are
 And nothing else matters
 Never opened myself this way
 Life is ours, we live it our way
 All these words, I don't just say
 And nothing else matters
 Trust I seek and I find in you
 Every day for us something new
 Open mind for a different view
 And nothing else matters
 Never cared for what they do
 Never cared for what they know
 But I know"]
```



8.3 random.choices() in Python 3

This is a very useful function[6]: weights (or probabilities) can be added.

(I used it in my Markov chains example (8.2).)

For example:

```
#!/usr/bin/env python3
import random

for i in range(1000):
    print (random.choices("123", [25, 60, 15]))
```

Let's generate 1000 random numbers in 1..3 range:

```
$ python3 tst.py | sort | uniq -c
    234 ['1']
    613 ['2']
    153 ['3']
```

"1" is generated in 25% of cases, "2" in 60% and "3" in 15%. Well, almost.

Here is another use of it.

You know, when you send an email, the final destination is a server somewhere. But it may be irresponsible. So network engineers add additional servers, "relays", which can hold your email for some time.

For example, what is about gmail.com?

```
% dig gmail.com MX

...
```

[6]https://docs.python.org/3/library/random.html#random.choices

```
;; ANSWER SECTION:
gmail.com.              3600    IN      MX      5 gmail-smtp-in.l.google.com.
gmail.com.              3600    IN      MX      10 alt1.gmail-smtp-in.l.google.com.
gmail.com.              3600    IN      MX      20 alt2.gmail-smtp-in.l.google.com.
gmail.com.              3600    IN      MX      30 alt3.gmail-smtp-in.l.google.com.
gmail.com.              3600    IN      MX      40 alt4.gmail-smtp-in.l.google.com.
...
```

The first server is primary (marked with 5). Other 4 (alt...) are relays. They can hold emails for `user@gmail.com` if the main server is down. Of course, relays also can be down. So an MTA (Message transfer agent) tries to send an email via the first server in list, then via the second, etc. If all are down, MTA is waiting for some time (not infinitely).

See also: https://en.wikipedia.org/wiki/MX_record.

A number (5/10/20/30/40) is priority:

```
   MX records contain a preference indication that MUST be used in
   sorting if more than one such record appears (see below).  Lower
   numbers are more preferred than higher ones.  If there are multiple
   destinations with the same preference and there is no clear reason to
   favor one (e.g., by recognition of an easily reached address), then
   the sender-SMTP MUST randomize them to spread the load across
   multiple mail exchangers for a specific organization.
```

(https://tools.ietf.org/html/rfc5321)

Now if you want your MTA be polite, you can make it poke relays with some probability, unloading the main mail server. In any case, the internal network withing Google is way better than a link between you and any of these mail servers. And it would be OK to drop an email to any of these mail servers listed in MX records.

This is how a destination server can be chosen:

```
random.choices(range(4), weights=[1/5, 1/10, 1/20, 1/40])
```

I'm using reciprocal weights (1/x) because the lower priority, the higher probability it is to be chosen.

What if I want to send 100 emails to `someone@gmail.com`?

```
>>> [random.choices(range(4), weights=[1/5, 1/10, 1/20, 1/40])[0] for x in range(100)
    ]

[1, 1, 2, 1, 0, 2, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0,
    1, 1, 1, 0, 1, 0, 2, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 3, 0, 0, 2, 1, 0,
    0, 0, 0, 1, 2, 2, 1, 0, 0, 0, 1, 2, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0, 1, 0, 2, 1, 0,
    0, 2, 0, 0, 0, 3, 2, 0, 1, 2, 0, 1, 1, 3, 1, 1, 1, 1]
```

1000? (I'm using the `collections.Counter`[7] here for gathering statistics).

```
>>> Counter([random.choices(range(4), weights=[1/5, 1/10, 1/20, 1/40])[0] for x in
    range(1000)])

Counter({0: 535, 1: 268, 2: 129, 3: 68})
```

535 emails will be sent via the first (primary) mail server, 268/129/68 – via corresponding relays.

This is probably not how MTAs usually operates, but this is how it could be done.

[7]https://docs.python.org/3/library/collections.html#collections.Counter

Chapter 9

Combinatorics

9.1 Soldering a headphones cable

This is a real story: I tried to repair my headphone's cable, soldering 3 wires with minijack together. But which is left? Right? I couldn't even find ground wire. I asked myself, how many ways there are to solder 3 wires to 3-pin minijack? I could try them all and pick the combination that sounds best.

With Python's itertools module this is just:

```
import itertools

wires=["red", "green", "blue"]

for i in itertools.permutations(wires):
        print i
```

```
('red', 'green', 'blue')
('red', 'blue', 'green')
('green', 'red', 'blue')
('green', 'blue', 'red')
('blue', 'red', 'green')
('blue', 'green', 'red')
```

(Just 6 ways.)

What if there are 4 wires?

```
import itertools

wires=["red", "green", "blue", "yellow"]

for i in itertools.permutations(wires):
        print i
```

```
('red', 'green', 'blue', 'yellow')
('red', 'green', 'yellow', 'blue')
('red', 'blue', 'green', 'yellow')
('red', 'blue', 'yellow', 'green')
('red', 'yellow', 'green', 'blue')
('red', 'yellow', 'blue', 'green')
('green', 'red', 'blue', 'yellow')
('green', 'red', 'yellow', 'blue')
('green', 'blue', 'red', 'yellow')
('green', 'blue', 'yellow', 'red')
('green', 'yellow', 'red', 'blue')
('green', 'yellow', 'blue', 'red')
```

```
('blue', 'red', 'green', 'yellow')
('blue', 'red', 'yellow', 'green')
('blue', 'green', 'red', 'yellow')
('blue', 'green', 'yellow', 'red')
('blue', 'yellow', 'red', 'green')
('blue', 'yellow', 'green', 'red')
('yellow', 'red', 'green', 'blue')
('yellow', 'red', 'blue', 'green')
('yellow', 'green', 'red', 'blue')
('yellow', 'green', 'blue', 'red')
('yellow', 'blue', 'red', 'green')
('yellow', 'blue', 'green', 'red')
```

(24 ways.)

This is what is called *permutation* in combinatorics.

9.2 Vehicle license plate

There was a hit and run. And you're police officer. And this is what your only witness can say about the 4-digit license plate of the runaway vehicle:

```
There was 13: 1 and 3 together, I'm sure, but not sure where, 13 as first 2 digits,
   last 2 digits or in the middle.
And there was also 6, or maybe 8, or maybe even 9, not sure which, but one of them.
```

Combinatorics textbooks are abound with exercises like this: can you enumerate all possible 4-digit numbers constrained in this way?

```
import itertools

part1_list=["13"]
part2_list=["6", "8", "9"]
part3_list=["0", "1", "2", "3", "4", "5", "6", "7", "8", "9"]

for i in itertools.product(part1_list, part2_list, part3_list):
    for j in itertools.permutations(i):
        print "".join(j)
```

```
1360
1306
6130
6013

...

9139
9913
9139
9913
```

(180 numbers)

This is something you can query registered vehicle database with...

9.3 Forgotten password

You forgot a password, but this is what you remember: there was a name of your parent, or wife, or one of children. Also, someone's year of birth. And one punctuation character, which are so recommended in passwords. Can you enumerate all possible passwords?

```
import itertools

part1_list=["jake", "melissa", "oliver", "emily"]
part2_list=["1987", "1954", "1963"]
part3_list=["!","@","#","$","%","&","*","-","=","_","+",".",","]

for part1 in part1_list:
    for part2 in part2_list:
        for part3 in part3_list:
            l=[part1, part2, part3]
            for i in list(itertools.permutations(l)):
                print "".join(i)
```

```
jake1987!
jake!1987
1987jake!
1987!jake
!jake1987

...

1963emily,
1963,emily
,emily1963
,1963emily
```

(936 of them in total)

But nested for's are not aesthetically pleasing. They can be replaced with ”cartesian product” operation:

```
import itertools

part1_list=["jake", "melissa", "oliver", "emily"]
part2_list=["1987", "1954", "1963"]
part3_list=["!","@","#","$","%","&","*","-","=","_","+",".",","]

for l in itertools.product(part1_list, part2_list, part3_list):
    for i in list(itertools.permutations(l)):
        print "".join(i)
```

And this is a way to memorize it: the length of the final result equals to lengths of all input lists multiplied with each other (like ”product”).

```
import itertools

part1_list=["jake", "melissa", "oliver", "emily"] # 4 elements
part2_list=["1987", "1954", "1963"] # 3 elements
part3_list=["!","@","#","$","%","&","*","-","=","_","+",".",","] # 13 elements

for l in itertools.product(part1_list, part2_list, part3_list):
    print l
```

```
('jake', '1987', '!')
('jake', '1987', '@')
('jake', '1987', '#')
('jake', '1987', '$')
('jake', '1987', '%')
('jake', '1987', '&')
('jake', '1987', '*')

...
```

```
('emily', '1963', '*')
('emily', '1963', '-')
('emily', '1963', '=')
('emily', '1963', '_')
('emily', '1963', '+')
('emily', '1963', '.')
('emily', '1963', ',')
```

4*3*13=156, and this is a size of a list, to be permuted...

Now the new problem: some Latin characters may be uppercased, some are lowercased. I'll add another "cartesian product" operation to alter a final string in all possible ways:

```
import itertools, string

part1_list=["jake", "melissa", "oliver", "emily"]
part2_list=["1987", "1954", "1963"]
part3_list=["!","@","#","$","%","&","*","-","=","_","+",".",","]

for l in itertools.product(part1_list, part2_list, part3_list):
    for i in list(itertools.permutations(l)):
        s="".join(i)
        t=[]
        for char in s:
            if char.isalpha():
                t.append([string.lower(char), string.upper(char)])
            else:
                t.append([char])
        for q in itertools.product(*t):
            print "".join(q)
```

```
JAke1987!
JAkE1987!
JAKe1987!
JAKE1987!
jake!1987
jakE!1987
jaKe!1987
jaKE!1987

...

,1963eMIly
,1963eMIlY
,1963eMILy
,1963eMILY
,1963Emily
,1963EmilY
,1963EmiLy
```

(56160 passwords)

Now leetspeak[1] This is somewhat popular only among youngsters, but still, this is what people of all age groups do: replacing "o" with "0" in their passwords, "e" with "3", etc. Let's add this as well:

```
import itertools, string

part1_list=["jake", "melissa", "oliver", "emily"]
part2_list=["1987", "1954", "1963"]
part3_list=["!","@","#","$","%","&","*","-","=","_","+",".",","]
```

[1] urbandictionary.com

```
for l in itertools.product(part1_list, part2_list, part3_list):
    for i in list(itertools.permutations(l)):
        s="".join(i)
        t=[]
        for char in s:
            if char.isalpha():
                to_be_appended=[string.lower(char), string.upper(char)]
                if char.lower()=='e':
                    to_be_appended.append('3')
                elif char.lower()=='i':
                    to_be_appended.append('1')
                elif char.lower()=='o':
                    to_be_appended.append('0')
                t.append(to_be_appended)
            else:
                t.append([char])
        for q in itertools.product(*t):
            print "".join(q)
```

```
jake1987!
jakE1987!
jak31987!
jaKe1987!
jaKE1987!
jaK31987!

...

,1963EM1lY
,1963EM1Ly
,1963EM1LY
,19633mily
,19633milY
,19633miLy
```

(140400 passwords)

Obviously, you can't try all 140400 passwords on Facebook, Twitter or any other well-protected internet service. But this is a peace of cake to brute-force them all on password protected RAR-archive or feed them to John the Ripper, HashCat, Aircrack-ng, etc.

All the files: https://αβγ.ελ/current_tree/comb/password.

Now let's use combinations from itertools Python package.

Let's say, you remember that your password has maybe your name, maybe name of your wife, your year of birth, or her, and maybe couple of symbols like `!`, `$`, `^`.

```
import itertools

parts=["den", "xenia", "1979", "1985", "secret", "!", "$", "^"]

for i in range(1, 6): # 1..5
    for combination in itertools.combinations(parts, i):
        print "".join(combination)
```

Here we enumerate all combinations of given strings, 1-string combinations, then 2-, up to 5-string combinations. No string can appear twice.

```
...
denxenia1979
```

```
denxenia1985
denxeniasecret
denxenia!
denxenia$
denxenia^
den19791985
den1979secret
den1979!
...
xenia1985secret$^
xenia1985!$^
xeniasecret!$^
19791985secret!$
19791985secret!^
...
```

(218 passwords)

Now let's permute all string in all possible ways:

```
import itertools

parts=["den", "xenia", "1979", "1985", "secret", "!", "$", "^"]

for i in range(1, 6): # 1..5
    for combination in itertools.combinations(parts, i):
        for permutation in itertools.permutations(list(combination)):
            print "".join(permutation)
```

```
...
^den
xenia1979
1979xenia
xenia1985
1985xenia
xeniasecret
secretxenia
xenia!
!xenia
...
^!$1985secret
^!$secret1985
^$1985secret!
^$1985!secret
^$secret1985!
^$secret!1985
^$!1985secret
^$!secret1985
```

(8800 passwords)

And finally, let's alter all Latin characters in lower/uppercase ways and add leetspeek, as I did before:

```
import itertools, string

parts=["den", "xenia", "1979", "1985", "!", "$", "^"]

for i in range(1, 6): # 1..5
    for combination in itertools.combinations(parts, i):
        for permutation in itertools.permutations(list(combination)):
            s="".join(permutation)
            t=[]
            for char in s:
                if char.isalpha():
                    to_be_appended=[string.lower(char), string.upper(char)]
                    if char.lower()=='e':
                        to_be_appended.append('3')
                    elif char.lower()=='i':
                        to_be_appended.append('1')
                    elif char.lower()=='o':
                        to_be_appended.append('0')
                    t.append(to_be_appended)
                else:
                    t.append([char])
            for q in itertools.product(*t):
                print "".join(q)
```

```
...
dEnxenia
dEnxeniA
dEnxenIa
...
D3nx3N1a
D3nx3N1A
D3nXenia
D3nXeniA
D3nXenIa
...
^$1979!1985
^$19851979!
^$1985!1979
^$!19791985
^$!19851979
```

(1,348,657 passwords)

Again, you can’t try to crack remote server with so many attempts, but this is really possible for password-protected archive, known hash, etc...

9.4 Executable file watermarking/steganography using Lehmer code and factorial number system

The prison authorities' contribution consisted of forbidding prisoners under investigation to receive any clothing or food packages. Sages of jurisprudence maintained that two French rolls, five apples and a pair of old pants were enough to transmit any text into the prison — even a fragment of *Anna Karenina*. Such 'messages from the free world' — an invention of the inflamed minds of diligent bureaucrats — were effectively prevented. A regulation was issued that only money could be sent, and it had to be in round figures of ten, twenty, thirty, forty, or fifty rubles; thus, numbers could not be used to work out a new 'alphabet' of messages.

Varlam Shalamov — Kolyma Tales

In short: how to hide information not in objects, but in *order* of objects.

Almost any binary executable file has text strings like (these are from CRT code):

```
.rdata:0040D398 aR6002FloatingP:
.rdata:0040D398                 text "UTF-16LE", 'R6002',0Dh,0Ah
.rdata:0040D398                 text "UTF-16LE", '- floating point support not loaded
   ',0Dh,0Ah,0
.rdata:0040D3F2                 align 8
.rdata:0040D3F8 aR6008NotEnough:
.rdata:0040D3F8                 text "UTF-16LE", 'R6008',0Dh,0Ah
.rdata:0040D3F8                 text "UTF-16LE", '- not enough space for arguments',0
   Dh,0Ah,0
.rdata:0040D44C                 align 10h
.rdata:0040D450 aR6009NotEnough:
.rdata:0040D450                 text "UTF-16LE", 'R6009',0Dh,0Ah
.rdata:0040D450                 text "UTF-16LE", '- not enough space for environment
   ',0Dh,0Ah,0
.rdata:0040D4A8 aR6010AbortHasB:
.rdata:0040D4A8                 text "UTF-16LE", 'R6010',0Dh,0Ah
.rdata:0040D4A8                 text "UTF-16LE", '- abort() has been called',0Dh,0Ah
   ,0
.rdata:0040D4EE                 align 10h
.rdata:0040D4F0 aR6016NotEnough:
.rdata:0040D4F0                 text "UTF-16LE", 'R6016',0Dh,0Ah
.rdata:0040D4F0                 text "UTF-16LE", '- not enough space for thread data
   ',0Dh,0Ah,0
.rdata:0040D548 aR6017Unexpecte:
.rdata:0040D548                 text "UTF-16LE", 'R6017',0Dh,0Ah
.rdata:0040D548                 text "UTF-16LE", '- unexpected multithread lock error
   ',0Dh,0Ah,0
.rdata:0040D5A2                 align 8
.rdata:0040D5A8 aR6018Unexpecte:
.rdata:0040D5A8                 text "UTF-16LE", 'R6018',0Dh,0Ah
.rdata:0040D5A8                 text "UTF-16LE", '- unexpected heap error',0Dh,0Ah,0
.rdata:0040D5EA                 align 10h
```

Can we hide some information there? Not in string themselves, but in *order* of strings? Given the fact, that compiler doesn't guarantee at all, in which order the strings will be stored in object/executable file.

Let's say, we've got 26 text strings, and we can swap them how we want, because their order isn't important at all. All possible permutations of 26 objects is 26! = 403291461126605635584000000.

How much information can be stored here? $log_2 26! =\approx 88$, i.e., 88 bits or 11 bytes!

11 bytes of your data can be converted to a (big) number and back, OK.

What is next? Naive way is: enumerate all permutations of 26 objects, number each, find permutation of the number we've got and permute 26 text strings, store to file and that's it. But we can't iterate over 403291461126605635584000000 permutations.

This is where factorial number system [2] and Lehmer code [3] can be used. In short, for all my non-mathematically inclined readers, this is a way to find a permutation of specific number without use of any significant resources. And back: given specific permutation, we can find its number.

This piece of code I've copypasted from https://gist.github.com/lukmdo/7049748 and reworked it slightly:

```
# python 3.x

import math

def iter_perm(base, *rargs):
    """
    :type base: list
    :param rargs: range args [start,] stop[, step]
    :rtype: generator
    """
    if not rargs:
        rargs = [math.factorial(len(base))]
    for i in range(*rargs):
        yield perm_from_int(base, i)

def int_from_code(code):
    """
    :type code: list
    :rtype: int
    """
    num = 0
    for i, v in enumerate(reversed(code), 1):
        num *= i
        num += v

    return num

def code_from_int(size, num):
    """
    :type size: int
    :type num: int
    :rtype: list
    """
    code = []
    for i in range(size):
        num, j = divmod(num, size - i)
        code.append(j)

    return code

def perm_from_code(base, code):
    """
```

[2] https://en.wikipedia.org/wiki/Factorial_number_system
[3] https://en.wikipedia.org/wiki/Lehmer_code

```
    :type base: list
    :type code: list
    :rtype: list
    """

    perm = base.copy()
    for i in range(len(base) - 1):
        j = code[i]
        if i != i+j:
            print ("swapping %d, %d" % (i, i+j))
        perm[i], perm[i+j] = perm[i+j], perm[i]

    return perm

def perm_from_int(base, num):
    """
    :type base: list
    :type num: int
    :rtype: list
    """
    code = code_from_int(len(base), num)
    print ("Lehmer code=", code)
    return perm_from_code(base, code)

def code_from_perm(base, perm):
    """
    :type base: list
    :type perm: list
    :rtype: list
    """

    p = base.copy()
    n = len(base)
    pos_map = {v: i for i, v in enumerate(base)}

    w = []
    for i in range(n):
        d = pos_map[perm[i]] - i
        w.append(d)

        if not d:
            continue
        t = pos_map[perm[i]]
        pos_map[p[i]], pos_map[p[t]] = pos_map[p[t]], pos_map[p[i]]
        p[i], p[t] = p[t], p[i]

    return w

def int_from_perm(base, perm):
    """
    :type base: list
    :type perm: list
    :rtype: int
    """
    code = code_from_perm(base, perm)
    return int_from_code(code)

def bin_string_to_number(s):
    rt=0
    for i, c in enumerate(s):
        rt=rt<<8
        rt=rt+ord(c)
    return rt

def number_to_bin_string(n):
    rt=""
    while True:
        r=n & 0xff
        rt=rt+chr(r)
        n=n>>8
        if n==0:
            break
    return rt[::-1]

s="HelloWorld"
print ("s=", s)
num=bin_string_to_number (s)
print ("num=", num)
perm=perm_from_int(list(range(26)), num)
print ("permutation/order=", perm)

num2=int_from_perm(list(range(26)), [14, 17, 9, 19, 11, 16, 23, 0, 2, 13, 20, 18, 21,
     24, 10, 1, 22, 4, 7, 6, 15, 12, 5, 3, 8, 25])
print ("recovered num=", num2)
s2=number_to_bin_string(num2)
print ("recovered s=", s2)
```

I'm encoding a "HelloWorld" binary string (in fact, any 11 bytes can be used) into a number. Number is then converted into Lehmer code. Then the `perm_from_code()` function permute initial *order* according to the input Lehmer code:

```
s= HelloWorld
num= 341881320659934023674980
Lehmer code= [14, 16, 7, 16, 7, 11, 17, 7, 1, 4, 10, 7, 9, 11, 6, 2, 6, 1, 2, 4, 0,
   0, 0, 0, 0, 0]
swapping 0, 14
swapping 1, 17
swapping 2, 9
swapping 3, 19
swapping 4, 11
swapping 5, 16
swapping 6, 23
swapping 7, 14
swapping 8, 9
swapping 9, 13
swapping 10, 20
swapping 11, 18
swapping 12, 21
swapping 13, 24
swapping 14, 20
swapping 15, 17
swapping 16, 22
swapping 17, 18
swapping 18, 20
swapping 19, 23
permutation/order= [14, 17, 9, 19, 11, 16, 23, 0, 2, 13, 20, 18, 21, 24, 10, 1, 22,
   4, 7, 6, 15, 12, 5, 3, 8, 25]
```

This is it: [14, 17, 9, 19, 11, 16, 23, 0, 2, 13, 20, 18, 21, 24, 10, 1, 22, 4, 7, 6, 15, 12, 5, 3, 8, 25]. First put 14th string, then 17s string, then 9th one, etc.

Now you've got a binary file from someone and want to read watermark from it. Get an order of strings from it and convert it back to binary string:

```
recovered num= 341881320659934023674980
recovered s= HelloWorld
```

If you have more text strings (not unusual for most executable files), you can encode more.

100 strings: $log_2(100!) \approx 524 bits \approx 65 bytes$.

1000 strings: $log_2(1000!) \approx 8529 bits \approx 1066 bytes$. You can store some text here!

How would you force a C/C++ compiler to make specific order of text strings? This is crude, but workable:

```
char blob[]="hello1\0hello2\0";
char *msg1=blob;
char *msg2=blob+8;

printf ("%s\n", msg1);
printf ("%s\n", msg2);
```

These text strings can be even aligned on 16-byte border.

... or they can be placed into .s/.asm assembly file and compiled into .o/.obj and then linked to your program.

... or you can swap text strings in already compiled executable and correct their addresses in corresponding instructions. If an executable file is not packed/obfuscated, this is possible.

Aside of order of text strings, you can try to hack a linker and reorder object files in the final executable. Of course, no one cares about its order. And go figure out, what is hidden there.

Surely, hidden data can be encrypted, checksum or MAC[4] can be added, etc.

Other ideas to consider: reorder functions inside within a module and fix all addresses, reorder basic blocks within a function, register allocator hacking, etc.

Links I find helpful in understanding factorial number system and Lehmer code, aside of Wikipedia:

- https://gist.github.com/lukmdo/7049748
- https://github.com/scmorse/permutils/blob/master/src/permutils.js
- http://2ality.com/2013/03/permutations.html
- http://www.keithschwarz.com/interesting/code/factoradic-permutation/FactoradicPermutation
- https://en.wikipedia.org/wiki/Mixed_radix
- https://en.wikipedia.org/wiki/Factorial_number_system

Also, this is what is called a *rank of permutation*: https://reference.wolfram.com/language/Combinatorica/ref/RankPermutation.html.

Further reading: Jorg Arndt — Matters Computational / Ideas, Algorithms, Source Code [5], 10.1 — Factorial representations of permutations.

Other ideas: encode information in order of files on your hard-disk or SSD. Or inside of a tar-archive.

9.5 De Bruijn sequences; leading/trailing zero bits counting

9.5.1 Introduction

Let's imagine there is a very simplified code lock accepting 2 digits, but it has no "enter" key, it just checks 2 last entered digits. Our task is to brute force each 2-digit combination. Naïve method is to try 00, 01, 02 ... 99. That require 2*100=200 key pressings. Will it be possible to reduce number of key pressings during brute-force? It is indeed so, with the help of De Bruijn sequences. We can generate them for the code lock, using Wolfram Mathematica:

```
In[]:= DeBruijnSequence[{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, 2]
Out[]= {6, 8, 6, 5, 4, 3, 2, 1, 7, 8, 7, 1, 1, 0, 9, 0, 8, 0, 6, 6, \
0, 5, 5, 0, 4, 4, 0, 3, 3, 0, 2, 7, 2, 2, 0, 7, 7, 9, 8, 8, 9, 9, 7, \
```

[4]Message authentication code
[5]https://www.jjj.de/fxt/fxtbook.pdf

```
0, 0, 1, 9, 1, 8, 1, 6, 1, 5, 1, 4, 1, 3, 7, 3, 1, 2, 9, 2, 8, 2, 6, \
2, 5, 2, 4, 7, 4, 2, 3, 9, 3, 8, 3, 6, 3, 5, 7, 5, 3, 4, 9, 4, 8, 4, \
6, 7, 6, 4, 5, 9, 5, 8, 5, 6, 9}
```

The result has exactly 100 digits, which is 2 times less than our initial idea can offer. By scanning visually this 100-digits array, you'll find any number in 00..99 range. All numbers are overlapped with each other: second half of each number is also first half of the next number, etc.

Here is another. We need a sequence of binary bits with all 3-bit numbers in it:

```
In[]:= DeBruijnSequence[{0, 1}, 3]
Out[]= {1, 0, 1, 0, 0, 0, 1, 1}
```

Sequence length is just 8 bits, but it has all binary numbers in 000..111 range. You may visually spot 000 in the middle of sequence. 111 is also present: two first bits of it at the end of sequence and the last bit is in the beginning. This is so because De Bruijn sequences are cyclic.

There is also visual demonstration: http://demonstrations.wolfram.com/DeBruijnSequences/.

9.5.2 Trailing zero bits counting

In the Wikipedia article about De Bruijn sequences we can find:

> The symbols of a De Bruijn sequence written around a circular object (such as a wheel of a robot) can be used to identify its angle by examining the n consecutive symbols facing a fixed point.

Indeed: if we know De Bruijn sequence and we observe only part of it (any part), we can deduce exact position of this part within sequence.

Let's see, how this feature can be used.

Let's say, there is a need to detect position of input bit within 32-bit word. For 0x1, the algorithm should report 1. 2 for 0x2. 3 for 0x4. And 31 for 0x80000000.

The result is in 0..31 range, so the result can be stored in 5 bits.

We can construct binary De Bruijn sequence for all 5-bit numbers:

```
In[]:= tmp = DeBruijnSequence[{0, 1}, 5]
Out[]= {1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0,
     1, 0, 0, 0, 0, 0}

In[]:= BaseForm[FromDigits[tmp, 2], 16]
Out[]:= e6bec520
```

Let's also recall that division some number by 2^n number is the same thing as shifting it by n bits. So if you divide 0xe6bec520 by 1, the result is not shifted, it is still the same. If if divide 0xe6bec520 by 4 (2^2), the result is shifted by 2 bits. We then take result and isolate lowest 5 bits. This result is unique number for each input. Let's shift 0xe6bec520 by all possible count number, and we'll get all possible last 5-bit values:

```
In[]:= Table[BitAnd[BitShiftRight[FromDigits[tmp, 2], i], 31], {i, 0, 31}]
Out[]= {0, 16, 8, 4, 18, 9, 20, 10, 5, 2, 17, 24, 12, 22, 27, 29, \
30, 31, 15, 23, 11, 21, 26, 13, 6, 19, 25, 28, 14, 7, 3, 1}
```

The table has no duplicates:

```
In[]:= DuplicateFreeQ[%]
Out[]= True
```

Using this table, it's easy to build a *magic* table. OK, now working C example:

```
#include <stdint.h>
#include <stdio.h>
```

```
int magic_tbl[32];

// returns single bit position counting from LSB
// not working for i==0
int bitpos (uint32_t i)
{
        return magic_tbl[(0xe6bec520/i) & 0x1F];
};

int main()
{
        // construct magic table
        // may be omitted in production code
        for (int i=0; i<32; i++)
                magic_tbl[(0xe6bec520/(1<<i)) & 0x1F]=i;

        // test
        for (int i=0; i<32; i++)
        {
                printf ("input=0x%x, result=%d\n", 1<<i, bitpos (1<<i));
        };
};
```

Here we feed our bitpos() function with numbers in 0..0x80000000 range and we got:

```
input=0x1, result=0
input=0x2, result=1
input=0x4, result=2
input=0x8, result=3
input=0x10, result=4
input=0x20, result=5
input=0x40, result=6
input=0x80, result=7
input=0x100, result=8
input=0x200, result=9
input=0x400, result=10
input=0x800, result=11
input=0x1000, result=12
input=0x2000, result=13
input=0x4000, result=14
input=0x8000, result=15
input=0x10000, result=16
input=0x20000, result=17
input=0x40000, result=18
input=0x80000, result=19
input=0x100000, result=20
input=0x200000, result=21
input=0x400000, result=22
input=0x800000, result=23
input=0x1000000, result=24
input=0x2000000, result=25
input=0x4000000, result=26
input=0x8000000, result=27
input=0x10000000, result=28
input=0x20000000, result=29
input=0x40000000, result=30
input=0x80000000, result=31
```

The bitpos() function actually counts trailing zero bits, but it works only for input values where only one bit is set. To make it more practical, we need to devise a method to drop all leading bits except of the last one. This method is very simple and well-known:

```
input & (-input)
```

This bit twiddling hack can solve the job. Feeding 0x11 to it, it will return 0x1. Feeding 0xFFFF0000, it will return 0x10000. In other words, it leaves lowest significant bit of the value, dropping all others.

It works because negated value in two's complement environment is the value with all bits flipped but also 1 added (because there is a zero in the middle of ring). For example, let's take 0xF0. -0xF0 is 0x10 or 0xFFFFFF10. ANDing 0xF0 and 0xFFFFFF10 will produce 0x10.

Let's modify our algorithm to support true trailing zero bits count:

```
#include <stdint.h>
#include <stdio.h>

int magic_tbl[32];

// not working for i==0
int tzcnt (uint32_t i)
{
        uint32_t a=i & (-i);
        return magic_tbl[(0xe6bec520/a) & 0x1F];
};

int main()
{
        // construct magic table
        // may be omitted in production code
        for (int i=0; i<32; i++)
                magic_tbl[(0xe6bec520/(1<<i)) & 0x1F]=i;

        // test:
        printf ("%d\n", tzcnt (0xFFFF0000));
        printf ("%d\n", tzcnt (0xFFFF0010));
};
```

It works!

```
16
4
```

But it has one drawback: it uses division, which is slow. Can we just multiplicate De Bruijn sequence by the value with the bit isolated instead of dividing sequence? Yes, indeed. Let's check in Mathematica:

```
In[]:= BaseForm[16^^e6bec520*16^^80000000, 16]
Out[]:= 0x735f629000000000
```

The result is just too big to fit in 32-bit register, but can be used. MUL/IMUL instruction 32-bit x86 CPUs stores 64-bit result into two 32-bit registers pair, yes. But let's suppose we would like to make portable code which will work on any 32-bit architecture. First, let's again take a look on De Bruijn sequence Mathematica first produced:

```
In[]:= tmp = DeBruijnSequence[{0, 1}, 5]
Out[]= {1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, \
0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0}
```

There is exactly 5 bits at the end which can be dropped. The "magic" constant will be much smaller:

```
In[]:= BaseForm[BitShiftRight[FromDigits[tmp, 2], 5], 16]
Out[]:=0x735f629
```

The "magic" constant is now "divided by 32 (or 1»5)". This mean that the result of multiplication of some value with one isolated bit by new magic number will also be smaller, so the bits we need will be stored at the high 5 bits of the result.

De Bruijn sequence is not broken after 5 lowest bits dropped, because these zero bits are ”relocated” to the start of the sequence. Sequence is cyclic after all.

```
#include <stdint.h>
#include <stdio.h>

int magic_tbl[32];

// not working for i==0
int tzcnt (uint32_t i)
{
        uint32_t a=i & (-i);
        // 5 bits we need are stored in 31..27 bits of product, shift and isolate them after
           multiplication:
        return magic_tbl[((0x735f629*a)>>27) & 0x1F];
};

int main()
{
        // construct magic table
        // may be omitted in production code
        for (int i=0; i<32; i++)
                magic_tbl[(0x735f629<<i >>27) & 0x1F]=i;

        // test:
        printf ("%d\n", tzcnt (0xFFFF0000));
        printf ("%d\n", tzcnt (0xFFFF0010));
};
```

9.5.3 Leading zero bits counting

This is almost the same task, but most significant bit must be isolated instead of lowest. This is typical algorithm for 32-bit integer values:

```
x |= x >> 1;
x |= x >> 2;
x |= x >> 4;
x |= x >> 8;
x |= x >> 16;
```

For example, 0x100 becomes 0x1ff, 0x1000 becomes 0x1fff, 0x20000 becomes 0x3ffff, 0x12340000 becomes 0x1fffffff. It works because all 1 bits are gradually propagated towards the lowest bit in 32-bit number, while zero bits at the left of most significant 1 bit are not touched.

It’s possible to add 1 to resulting number, so it will becomes 0x2000 or 0x20000000, but in fact, since multiplication by magic number is used, these numbers are very close to each other, so there are no error.

This example I used in my reverse engineering exercise from 15-Aug-2015: https://yurichev.com/blog/2015-aug-18/.

```
int v[64]=
        { -1,31, 8,30, -1, 7,-1,-1, 29,-1,26, 6, -1,-1, 2,-1,
          -1,28,-1,-1, -1,19,25,-1, 5,-1,17,-1, 23,14, 1,-1,
           9,-1,-1,-1, 27,-1, 3,-1, -1,-1,20,-1, 18,24,15,10,
          -1,-1, 4,-1, 21,-1,16,11, -1,22,-1,12, 13,-1, 0,-1 };

int LZCNT(uint32_t x)
{
    x |= x >> 1;
    x |= x >> 2;
    x |= x >> 4;
    x |= x >> 8;
    x |= x >> 16;
    x *= 0x4badf0d;
    return v[x >> 26];
}
```

This piece of code I took from here. It is slightly different: the table is twice bigger, and the function returns -1 if input value is zero. The magic number I found using just brute-force, so the readers will not be able to google it, for the sake of exercise. (By the way, I've got 12,665,720 magic numbers which can serve this purpose. This is about 0.294

The code is tricky after all, and the moral of the exercise is that practicing reverse engineer sometimes may just observe input/outputs to understand code's behaviour instead of diving into it.

9.5.4 Performance

The algorithms considered are probably fastest known, they has no conditional jumps, which is very good for CPUs starting at RISCs. Newer CPUs has LZCNT and TZCNT instructions, even 80386 had BSF/BSR instructions which can be used for this: https://en.wikipedia.org/wiki/Find_first_set. Nevertheless, these algorithms can be still used on cheaper RISC CPUs without specialized instructions.

9.5.5 Applications

Number of leading zero bits is binary logarithm of value: 11.

These algorithms are also extensively used in chess engines programming, where each piece is represented as 64-bit bitmask (chess board has 64 squares): http://chessprogramming.wikispaces.com/BitScan.

There are more: https://en.wikipedia.org/wiki/Find_first_set#Applications.

9.5.6 Generation of De Bruijn sequences

De Bruijn graph is a graph where all values are represented as vertices (or nodes) and each edge (or link) connects two nodes which can be "overlapped". Then we need to visit each edge only once, this is called *eulerian path*. It is like the famous *task of seven bridges of Königsberg*: traveller must visit each bridge only once.

There are also simpler algorithms exist: https://en.wikipedia.org/wiki/De_Bruijn_sequence#Algorithm.

9.5.7 Other articles

At least these are worth reading: http://supertech.csail.mit.edu/papers/debruijn.pdf, http://alexandria.tue.nl/repository/books/252901.pdf, Wikipedia Article about De Bruijn sequences.

https://chessprogramming.wikispaces.com/De+Bruijn+sequence, https://chessprogramming.wikispaces.com/De+Bruijn+Sequence+Generator.

Chapter 10

Galois Fields, GF(2)

10.1 Remainder division: yet another explanation of CRC

10.1.1 What is wrong with checksum?

If you just sum up values of several bytes, two bit flips (increment one bit and decrement another bit) can give the same checksum. No good.

10.1.2 Division by prime

You can represent a file of buffer as a (big) number, then to divide it by prime. The remainder is then very sensitive to bit flips. For example, a prime 0x10015 (65557).

Wolfram Mathematica:

```
In[]:= divisor=16^^10015
Out[]= 65557

In[]:= BaseForm[Mod[16^^abcdef1234567890, divisor],16]
Out[]= d8c1

In[]:= BaseForm[Mod[16^^abcdef0234567890, divisor],16]
Out[]= bd31

In[]:= BaseForm[Mod[16^^bbcdef1234567890, divisor],16]
Out[]= 382b

In[]:= BaseForm[Mod[16^^abcdee1234567890, divisor],16]
Out[]= 1fd6

In[]:= BaseForm[Mod[16^^abcdef0234567891, divisor],16]
Out[]= bd32
```

This is what is called "avalanche effect" in cryptography: one bit flip of input can affect many bits of output. Go figure out which bits must be also flipped to preserve specific remainder.

You can build such a divisor in hardware, but it would require at least one adder or subtractor, you will have a carry-ripple problem in simple case, or you would have to create more complicated circuit.

10.1.3 (Binary) long division

Binary long division is in fact simpler then the paper-n-pencil algorithm taught in schools.

The algorithm is:

- 1) Allocate some "tmp" variable and copy dividend to it.
- 2) Pad divisor by zero bits at left so that MSB of divisor is at the place of MSB of the value in tmp.

- 3) If the divisor is larger than tmp or equal, subtract divider from tmp and add 1 bit to the quotient. If the divisor is smaller than tmp, add 0 bit to the quotient.
- 4) Shift divisor right. If the divisor is 0, stop. Remainder is in tmp.
- 5) Goto 3

The following piece of code I've copypasted from somewhere:

```
unsigned int divide(unsigned int dividend, unsigned int divisor)
{
        unsigned int tmp = dividend;
        unsigned int denom = divisor;
        unsigned int current = 1;
        unsigned int answer = 0;

        if (denom > tmp)
                return 0;

        if (denom == tmp)
                return 1;

        // align divisor:
        while (denom <= tmp)
        {
                denom = denom << 1;
                current = current << 1;
        }

        denom = denom >> 1;
        current = current >> 1;

        while (current!=0)
        {
                printf ("current=%d, denom=%d\n", current, denom);
                if (tmp >= denom)
                {
                        tmp -= denom;
                        answer |= current;
                }
                current = current >> 1;
                denom = denom >> 1;
        }
        printf ("tmp/remainder=%d\n", tmp); // remainder!
        return answer;
}
```

(https://αβγ.ελ/current_tree/GF2/CRC/div.c)

Let's divide 1234567 by 813 and find remainder:

```
current=1024, denom=832512
current=512, denom=416256
current=256, denom=208128
current=128, denom=104064
current=64, denom=52032
current=32, denom=26016
current=16, denom=13008
current=8, denom=6504
current=4, denom=3252
current=2, denom=1626
current=1, denom=813
tmp/remainder=433
1518
```

10.1.4 (Binary) long division, version 2

Now let's say, you only need to compute a remainder, and throw away a quotient. Also, maybe you work on some kind BigInt values and you've got a function like `get_next_bit()` and that's it.

What we can do: tmp value will be shifted at each iteration, while divisor is not:

```
uint8_t *buf;
int buf_pos;
int buf_bit_pos;

int get_bit()
{
        if (buf_pos==-1)
                return -1; // end

        int rt=(buf[buf_pos] >> buf_bit_pos) & 1;
        if (buf_bit_pos==0)
        {
                buf_pos--;
                buf_bit_pos=7;
        }
        else
                buf_bit_pos--;
        return rt;
};

uint32_t remainder_arith(uint32_t dividend, uint32_t divisor)
{
        buf=(uint8_t*)&dividend;
        buf_pos=3;
        buf_bit_pos=7;

        uint32_t tmp=0;

        for(;;)
        {
                int bit=get_bit();
                if (bit==-1)
                {
                        printf ("exit. remainder=%d\n", tmp);
                        return tmp;
                };

                tmp=tmp<<1;
                tmp=tmp|bit;

                if (tmp>=divisor)
                {
                        printf ("%d greater or equal to %d\n", tmp, divisor);
                        tmp=tmp-divisor;
                        printf ("new tmp=%d\n", tmp);
                }
                else
                        printf ("tmp=%d, can't subtract\n", tmp);
        };
}
```

(https://αβγ.ελ/current_tree/GF2/CRC/div_both.c)

Let's divide 1234567 by 813 and find remainder:

```
tmp=0, can't subtract
```

```
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=0, can't subtract
tmp=1, can't subtract
tmp=2, can't subtract
tmp=4, can't subtract
tmp=9, can't subtract
tmp=18, can't subtract
tmp=37, can't subtract
tmp=75, can't subtract
tmp=150, can't subtract
tmp=301, can't subtract
tmp=602, can't subtract
1205 greater or equal to 813
new tmp=392
tmp=785, can't subtract
1570 greater or equal to 813
new tmp=757
1515 greater or equal to 813
new tmp=702
1404 greater or equal to 813
new tmp=591
1182 greater or equal to 813
new tmp=369
tmp=738, can't subtract
1476 greater or equal to 813
new tmp=663
1327 greater or equal to 813
new tmp=514
1029 greater or equal to 813
new tmp=216
tmp=433, can't subtract
exit. remainder=433
```

10.1.5 Shortest possible introduction into GF(2)

There is a difference between digit and number. Digit is a symbol, number is a group of digits. 0 can be both digit and number.

Binary digits are 0 and 1, but a binary number can be any.

There are just two numbers in Galois Field (2): 0 and 1. No other numbers.

What practical would you do with just two numbers? Not much, but you can pack GF(2) numbers into some kind of structure or tuple or even array. Such structures are represented using polynomials. For example, CRC32 polynomial you can find in source code is 0x04C11DB7. Each bit represent a number in GF(2), not a digit. The 0x04C11DB7 polynomial is written as:

$x^{32} + x^{26} + x^{23} + x^{22} + x^{16} + x^{12} + x^{11} + x^{10} + x^{8} + x^{7} + x^{5} + x^{4} + x^{2} + x + 1$

Wherever x^n is present, that means, you have a bit at position n. Just x means, bit present at LSB. There is, however, bit at x^{32}, so the CRC32 polynomial has the size of 33 bits, bit the MSB is always 1 and is omitted in all algorithms.

It's important to say that unlike in algebra, GF(2) polynomials are never evaluated here. x is symbol is present merely as a convention. People represent GF(2) "structures" as polynomials to emphasize the fact that "numbers" are isolated from each other.

Now, subtraction and addition are the same operations in GF(2) and actually works as XOR. This is present in many tutorials, so I'll omit this here.

Also, by convention, whenever you compare two numbers in GF(2), you only compare two most significant bits, and ignore the rest.

10.1.6 CRC32

Now we can take the binary division algorithm and change it a little:

```
uint32_t remainder_GF2(uint32_t dividend, uint32_t divisor)
{
        // necessary bit shuffling/negation to make it compatible with other CRC32
           implementations.
        // N.B.: input data is not an array, but a 32-bit integer, hence we need to swap
           endiannes.
        uint32_t dividend_negated_swapped = ~swap_endianness32(bitrev32(dividend));
        buf=(uint8_t*)&dividend_negated_swapped;
        buf_pos=3;
        buf_bit_pos=7;

        uint32_t tmp=0;

        // process 32 bits from the input + 32 zero bits:
        for(int i=0; i<32+32; i++)
        {
                int bit=get_bit();
                int shifted_bit=tmp>>31;

                // fetch next bit:
                tmp=tmp<<1;
                if (bit==-1)
                {
                        // no more bits, but continue, we fetch 32 more zero bits.
                        // shift left operation set leftmost bit to zero.
                }
                else
                {
                        // append next bit at right:
                        tmp=tmp|bit;
                };

                // at this point, tmp variable/value has 33 bits: shifted_bit + tmp
                // now take the most significant bit (33th) and test it:
                // 33th bit of polynomial (not present in "divisor" variable is always 1
                // so we have to only check shifted_bit value
                if (shifted_bit)
                {
                        // use only 32 bits of polynomial, ignore 33th bit, which is always
                           1:
                        tmp=tmp^divisor;
                };
        };
        // bit shuffling/negation for compatibility once again:
        return ~bitrev32(tmp);
}
```

(https://αβγ.ελ/current_tree/GF2/CRC/div_both.c)

And voila, this is the function which computes CRC32 for the input 32-bit value.

There are only 3 significant changes:

- XOR instead of minus.

- Only MSB is checked during comparison. But the MSB of all CRC polynomials is always 1, so we only need to check MSB (33th bit) of the tmp variable.
- There are 32+32=64 iterations instead of 32. As you can see, only MSB of tmp affects the whole behaviour of the algorithm. So when tmp variable is filled by 32 bits which never affected anything so far, we need to ”blow out” all these bits through 33th bit of tmp variable to get correct remainder (or CRC32 sum).

All the rest algorithms you can find on the Internet are optimized version, which may be harder to understand. No algorithms used in practice “blows” anything “out” due to optimization. Many practical algorithms are either bytewise (process input stream by bytes, not by bits) or table-based.

My goal was to write two functions, as similar to each other as possible, to demonstrate the difference.

So the CRC value is in fact remainder of division of input date by CRC polynomial in GF(2) environment. As simple as that.

10.1.7 Rationale

Why use such an unusual mathematics? The answer is: many GF(2) operations can be done using bit shifts and XOR, which are very cheap operations.

Electronic circuit for CRC generator is extremely simple, it consists of only shift register and XOR gates. This one is for CRC16:

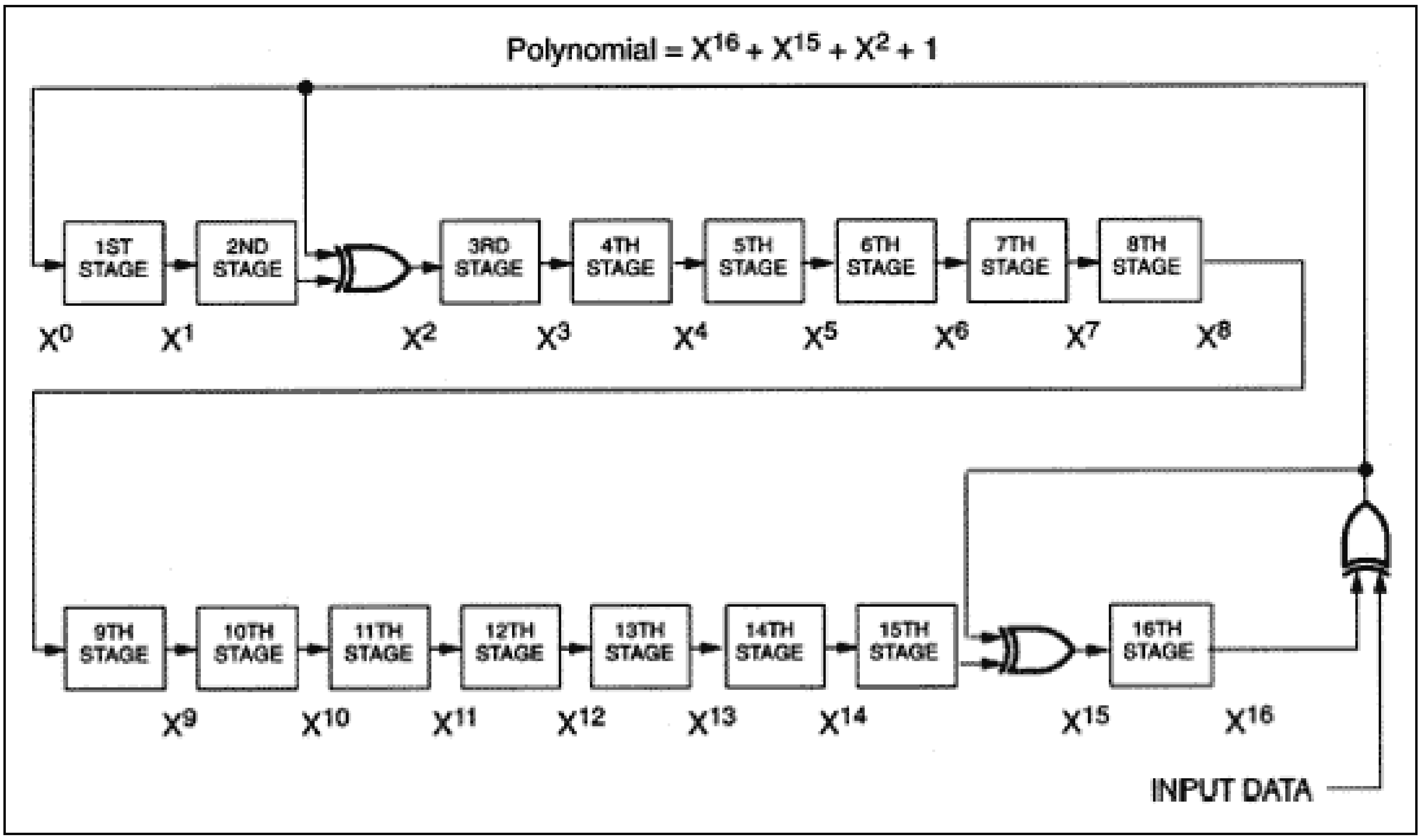


Figure 10.1:

(The source of image: https://olimex.wordpress.com/2014/01/10/weekend-programming-challenge-week-39-crc-16/)

Only 3 XOR gates are present aside of shift register.

The following page has animation: https://en.wikipedia.org/wiki/Computation_of_cyclic_redundancy_checks.

It can be implemented maybe even using vacuum tubes.

And the task is not to compute remainder according to rules of arithmetics, but rather to detect errors.

Compare this to a division circuit with at least one binary adder/subtractor, which will have carry-ripple problem. On the other hand, addition over GF(2) has no carries, hence, this problem absent.

10.1.8 Further reading

These documents I've found interesting/helpful:

- http://www.ross.net/crc/download/crc_v3.txt
- https://www.kernel.org/doc/Documentation/crc32.txt
- http://web.archive.org/web/20161220015646/http://www.hackersdelight.org/crc.pdf

10.2 Multiplication: Galois/Counter Mode

GCM is a way of encrypting data plus MAC. Its central operation is GF(2) or GF(2^{128}) multiplication.

Here is a Python code with my experiments around GF(2) multiplication operation. It's reworked version I stolen here. See comments.

```
#!/usr/bin/env python3

R =  0xE1000000000000000000000000000000

# I often use reverseBits() here.
# rationale: identity element would be represented as decimal number 1, not as 0x80....00

# yes, both are XORs:
def gf2_add(x, y):
    return x^y

def gf2_sub(x, y):
    return x^y

def gf128_mul(x, y, R):
    z = 0
    for i in range(128-1, -1, -1):

        if (y >> i) & 1:
            z=gf2_add(z, x)

        # shift and also reduce by R if overflow detected
        # IOW, keep x smaller than R or modulo R
        if x & 1:
            x = gf2_sub(x >> 1, R)
        else:
            x = x >> 1

    return z

def gf2_pow_2(x, R):
    return gf128_mul(x, x, R)

def is_odd(n):
    return n&1==1

# almost as in https://en.wikipedia.org/wiki/Exponentiation_by_squaring
def gf2_pow(x, n, R):
    if n==1:
         return x
    if is_odd(n):
        return gf128_mul(x, gf2_pow(gf2_pow_2(x, R), (n-1)//2, R), R)
    else:
        return gf2_pow(gf2_pow_2(x, R), n//2, R)

# return $x^{2^{128}-2}$
# AKA reciprocal AKA modulo inverse
def gf2_inv(x, R):
    rslt=reverseBits(1, 128) # init to 1

    for i in range(128-1):
        rslt=gf128_mul(rslt, x, R)
        x=gf2_pow_2(x, R)

    return gf2_pow_2(rslt, R)

# simpler version:
def gf2_inv_v2(x, R):
    return gf2_pow(x, 2**128-2, R)

# https://math.stackexchange.com/questions/943417/square-root-for-galois-fields-gf2m
# https://crypto.stackexchange.com/questions/17988/algorithm-for-computing-square-roots-in-gf2n
# return $x^{2^{127}}$
def gf2_sqrt(x, R):
    for i in range(127):
        x=gf2_pow_2(x, R)

    return x

# simpler version:
def gf2_sqrt_v2(x, R):
    return gf2_pow(x, 2**127, R)

def gf128_div(N, D, R):
    # TODO: check for zero
    # N = numerator (dividend)
    # D = denominator (divisor)
    i1=gf2_inv(D, R)
    # also, test 2nd version:
    i2=gf2_inv_v2(D, R)
    assert i1==i2

    return gf128_mul(N, i1, R)
```

Full version: https://αβγ.ελ/current_tree/GF2/GCM/1.py

Thing to mention: since the polynomial used in AES-GCM is irreducible, it's like *prime*. Hence, $a^{-1} \equiv a^{m-2} \pmod{m}$.

10.2.1 Irreducibility

Since the AES-GCM polynomial is irreducible, any number within GF(2^{128}) with this defining polynomial has modulo inverse.

This is not true for modulo inverses for $\mathbb{Z}/n$ where $n = 2^x$ (when replacing division by multiplication: 7.3). For example, you can't find modulo inverse for even number in $\mathbb{Z}/(2^x)$, because any even number is not coprime to 2^x. (Compilers implement division by even numbers in different ways.) But any odd number is. So a modulo inverse exists for any odd number.

This is example, where modulo inverse exists for any $\mod p$, where p is prime:

```
#!/usr/bin/env python3
import math

# first prime less than $2^{32}$, IOW, closest
p=4294967291 # 0xfffffffb

```

```
# modulo inverse exist for any divisior, since any divisior<p is coprime to p, since p is prime
# for prime p, EulerPhi[p]=p-1
def test(divisor):
    # calculate modulo inverse: divisor^{p-2} (mod p)
    mod_inv=pow(divisor, p-2, p)

    i=12340
    r1=(i/divisor) % p
    r2=(i*mod_inv) % p
    assert r1==r2

test(2)
test(5)
test(10)
test(1234)
```

10.2.2 Multiplication mod p over $\mathbb{Z}$

And here is a multiplication function mod p, which acts just like the code for GF(2) above, but for $\mathbb{Z}$:

```
#!/usr/bin/env python3
import random, math

def ceil_binlog(x):
    return math.ceil(math.log(x, 2))

# just multiplication
def mul(x_in, y):
    x = x_in
    z = 0
    for i in range(ceil_binlog(y)):

        if (y >> i) & 1:
            z=z+x

        x = x << 1

    assert z==x_in*y # self-test
    return z

# multiplication modulo m
# another version: https://en.wikipedia.org/wiki/Modular_arithmetic#Example_implementations
def mul_mod(x_in, y_in, m):
    # hack/kludge. not practical. but for demonstration. keep initial x and y mod m:
    x = x_in % m
    y = y_in % m
    z = 0
    for i in range(ceil_binlog(y)):

        if (y >> i) & 1:
            # be sure z and x mod m
            assert z<m and x<m
            z=z+x # <- 'core' operation

        x = x << 1

        # if x overflown, keep it mod m
        if x > m:
            x = x - m
        assert x<m # be sure...
        # same for z
        if z > m:
            z = z - m
        assert z<m # be sure...

    assert z==(x_in*y_in) % m # self-test
    return z
```

Full version: https://αβγ.ελ/current_tree/GF2/GCM/mul.py

You may see similarities in two versions.

10.2.3 Further reading

Russ Cox: https://research.swtch.com/field.

10.2.4 Files

SageMath notebooks I used: https://αβγ.ελ/current_tree/GF2/GCM/files/

10.3 Important note

We've been taught in school that mathematics is all about counting objects (apples) or areas (geometry).

But there are algebras (GF) which can't count objects, however, have enough useful features to use them practically. GF is one of them.

Chapter 11

Logarithms

11.1 Introduction

11.1.1 Children's approach

When children argue about how big their favorite numbers are, they speaking about how many zeroes it has: "x has n zeroes!" "No, my y is bigger, it has $m > n$ zeroes!"

This is exactly notion of common (base 10) logarithm.

Googol (10^{100}) has 100 zeroes, so $log_{10}(googol) = 100$.

Let's take some big number, like 12th Mersenne prime:

Listing 11.1: Wolfram Mathematica

```
In[]:= 2^127 - 1
Out[]= 170141183460469231731687303715884105727
```

Wow, it's so big. How can we measure it in childish terms? How many digits it has? We can count using common (base 10) logarithm:

Listing 11.2: Wolfram Mathematica

```
In[]:= Log[10, 2^127 - 1] // N
Out[]= 38.2308
```

So it has 39 digits.

Another question, how may decimal digits 1024-bit RSA key has?

Listing 11.3: Wolfram Mathematica

```
In[]:= 2^1024
Out[]= 17976931348623159077293051907890247336179769789423065727343008\
115773267580550096313270847732240753602112011387987139335765878976881 4\
416622492847430639474124377767893424865485276302219601246094119453082 9\
520850057688381506823424628814739131105408272371633505106845862982399 4\
7245938479716304835356329624224137216

In[]:= Log10[2^1024] // N
Out[]= 308.255
```

309 decimal digits.

11.1.2 Scientists' and engineers' approach

Interestingly enough, scientists' and engineers' approach is not very different from children's. They are not interesting in noting each digit of some big number, they are usually interesting in three properties of some number: 1) sign; 2) first n digits (significand or mantissa); 3) exponent (how many digits the number has).

The common way to represent a real number in handheld calculators and FPUs is:

$$(sign)significand \times 10^{exponent} \tag{11.1}$$

For example:

$$-1.987126381 \times 10^{41} \tag{11.2}$$

It was common for scientific handheld calculators to use the first 10 digits of significand and ignore everything behind. Storing the whole number down to the last digit is 1) very expensive; 2) hardly useful.

The number in IEEE 754 format (most popular way of representing real numbers in computers) has these three parts, however, it has different base (2 instead of 10).

11.2 Logarithmic scale

11.2.1 In human perception

Logarithmic scale is very natural to human perceptions, including eyes. When you ask average human to judge on current lighting, he/she may use words like “dark”, “very dark”, “normal”, “twilight”, “bright”, “like on beach”. In human language, there are couple of steps between “dark” and “bright”, but luminous intensity may differ by several orders of magnitude. Old cheap “point-n-shoot” photo cameras also has scale expressed in natural human languages. But professional photo cameras also has logarithmic scales:

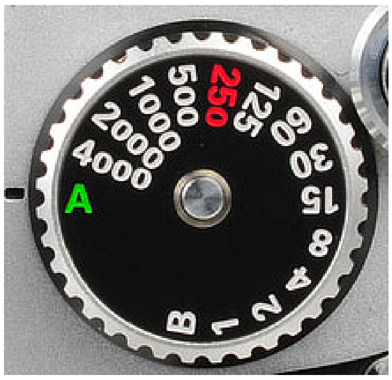


Figure 11.1: Shutter speed ($\frac{1}{x}$ of second) knob on photo camera

Another logarithmic scale familiar to anyone is decibel. Even on cheap mp3 players and smartphones, where the volume is measured in conventional percents, this scale is logarithmic, and the difference between 50% and 60% may be much larger in sound pressure terms.

Yet another familiar to anyone logarithmic scale is Richter magnitude scale [1]. The Richter scale is practical, because when people talk about earthquakes, they are not interesting in exact scientific values (in Joules or TNT equivalent), they are interesting in how bad damage is.

11.2.2 In electronics engineering

The loudspeakers are not perfect, so its output is non-linear in relation to input frequency. In other word, loudspeaker has different loudness at different frequency. It can be measured easily, and here is an example of plot of some speaker, I took it there: http://www.3dnews.ru/270838/page-3.html.

[1] https://en.wikipedia.org/wiki/Richter_magnitude_scale

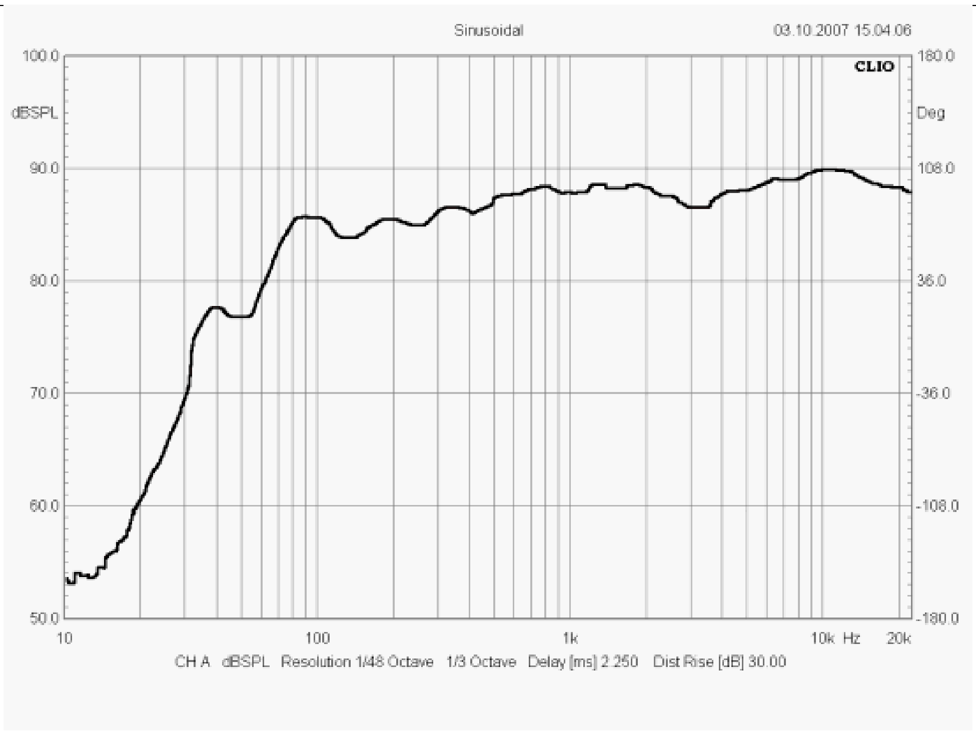


Figure 11.2: Frequency response (also known as *Bode plot*) of some loudspeaker

Both axis on this plot are logarithmic: y axis is loudness in decibel and x axis is frequency in Hertz. Needless to say, the typical loudspeaker has bass/medium speaker + tweeter (high frequency speaker). Some of more advanced loudspeaker has 3 speakers: bass, medium and tweeter. Or even more. And since the plot is logarithmic, each of these 2 or 3 speakers has their own part of plot, and these parts has comparable size. If the x axis would be linear instead of logarithmic, the main part of it would be occupied by frequency response of tweeter alone, because it has widest frequency range. While bass speaker has narrowest frequency range, it would have very thin part of the plot.

y axis (vertical) of the plot is also logarithmic (its value is shown in decibels). If this axis would be linear, the main part of it would be occupied by very loud levels of sound, while there would be thinnest line at the bottom reserved for normal and quiet level of sounds.

Both of that would make plot unusable and impractical. So both axis has logarithmic scale. In strict mathematics terms, the plot shown is called *log-log plot*, which means that both axis has logarithmic scale.

Summarizing, both electronics engineers and HiFi audio enthusiasts use these plots to compare quality of speakers. These plots are often used in loudspeakers reviews.

Some of speakers of USSR era (like Latvian Radiotehnika S-30 and S-90) had such plots right on the surface of speaker box, presumably, for marketing purposes:

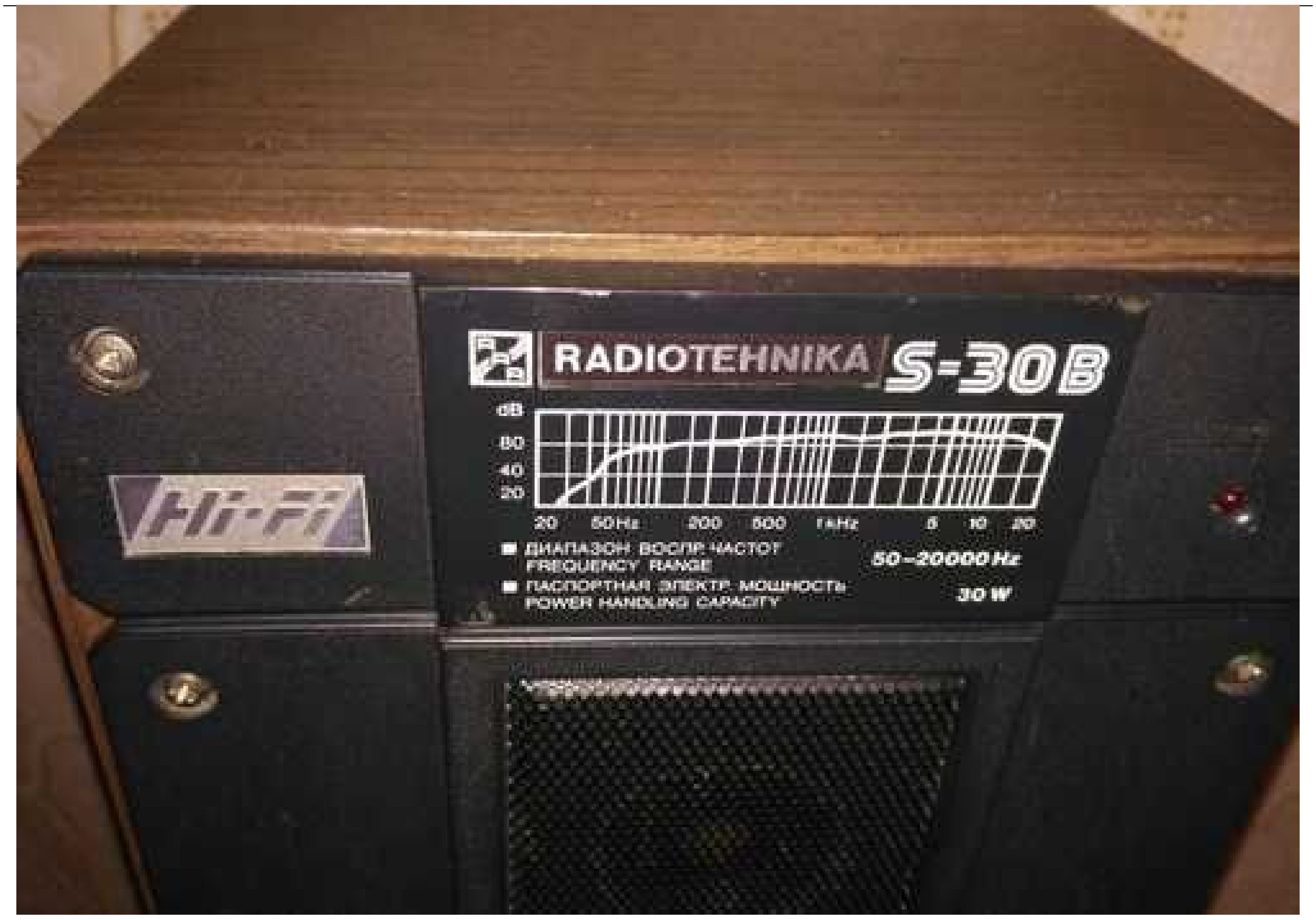


Figure 11.3: Radiotehnika S-30b

11.2.3 In IT

git, like any other VCS, can show a graph, how many changes each file got in each commit, for example:

```
$ git log --stat

...

commit 2fb3437fa753d59ba37f3d11c7253583d4b87c99
Author: Dennis Yurichev <dennis@yurichev.com>
Date:   Wed Nov 19 14:14:07 2014 +0200

    reworking `64-bit in 32-bit environment' part

 patterns/185_64bit_in_32_env/0.c                          |   6 --
 patterns/185_64bit_in_32_env/0_MIPS.s                     |   5 -
 patterns/185_64bit_in_32_env/0_MIPS_IDA.lst               |   5 -
 patterns/185_64bit_in_32_env/0_MSVC_2010_Ox.asm           |   5 -
 patterns/185_64bit_in_32_env/1.c                          |  20 ----
 patterns/185_64bit_in_32_env/1_GCC.asm                    |  27 ------
 patterns/185_64bit_in_32_env/1_MSVC.asm                   |  31 -------
 patterns/185_64bit_in_32_env/2.c                          |  16 ----
 patterns/185_64bit_in_32_env/2_GCC.asm                    |  41 ---------
 patterns/185_64bit_in_32_env/2_MSVC.asm                   |  32 -------
 patterns/185_64bit_in_32_env/3.c                          |   6 --
 patterns/185_64bit_in_32_env/3_GCC.asm                    |   6 --
 patterns/185_64bit_in_32_env/3_MSVC.asm                   |   8 --
 patterns/185_64bit_in_32_env/4.c                          |  11 ---
 patterns/185_64bit_in_32_env/4_GCC.asm                    |  35 -------
 patterns/185_64bit_in_32_env/4_MSVC.asm                  |  30 ------
 patterns/185_64bit_in_32_env/conversion/4.c              |   6 ++
 patterns/185_64bit_in_32_env/conversion/Keil_ARM_O3.s    |   4 +
 patterns/185_64bit_in_32_env/conversion/MSVC2012_Ox.asm  |   6 ++
 patterns/185_64bit_in_32_env/conversion/main.tex         |  48 ++++++++++
 patterns/185_64bit_in_32_env/main.tex                    | 127
    +------------------------

...
```

This scale is not logarithmical (I had a look into git internals), but this is exact place where logarithmical scale can be used. When software developer got such report, he/she don't interesting in exact numbers of lines changed/added/removed. He/she wants to see an outlook: which files got most changes/additions/removals, and which got less.

There is also a constraint: the space on the terminal is limited, so it's not possible to draw a minus or plus sign for each changed line of code.

Another example is Bitcoin client "signal reception strength", apparently, modeled after mobile phone signal indicator:

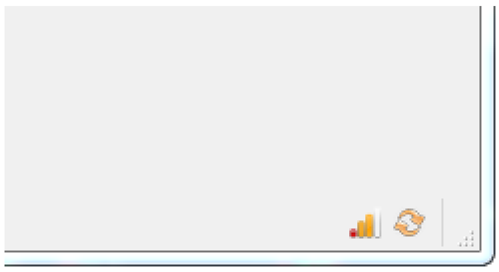


Figure 11.4: Bitcoin client

These bars indicating, how many connections client currently has. Let's imagine, client can support up to 1000 connections, but user is never interesting in precise number, all he/she wants to know is how good its link with Bitcoin network is. I don't know how Bitcoin calculates this, but I think one bar could indicate that client has only 1 connection, two bars — 2-5, three bars — up to 10-20, and four bars — anything bigger. This is also logarithmic scale. On contrary, if you divide 1000 by 4 even parts, and one bar will fired if you've got 250 connections, two bars if you've got 500, etc, this would make the indicator useless, such indicators are no better than simple "on/off" lamp.

11.2.4 Web 2.0

Sites like GitHub, Reddit, Twitter sometimes shows how long some event was ago, instead of precise date (at least in 2015). For example, Reddit may show date as "3 years ago", "11 months ago", "3 weeks ago", "1 day ago", "10 hours ago", etc, down to minutes and seconds. You wouldn't see "3 years and 11 hours ago". This is also logarithmic scale. When some event happens 10 months ago, users are typically not interesting in precision down to days and hours. When something happens 2 years ago, users usually not interesting in number of months and days in addition to these 2 years.

11.2.5 Errors of TeX

Here Donald Knuth uses logarithmic scale because he fixed many bugs at the beginning of TeX development, in 1982-1983. After that, in 1984-1985, bug count was dropped. So in order to show a nice linear graph with text labels, he uses logarithmic scale for time (vertical axis), but horizontal axis (for bug count) is linear.

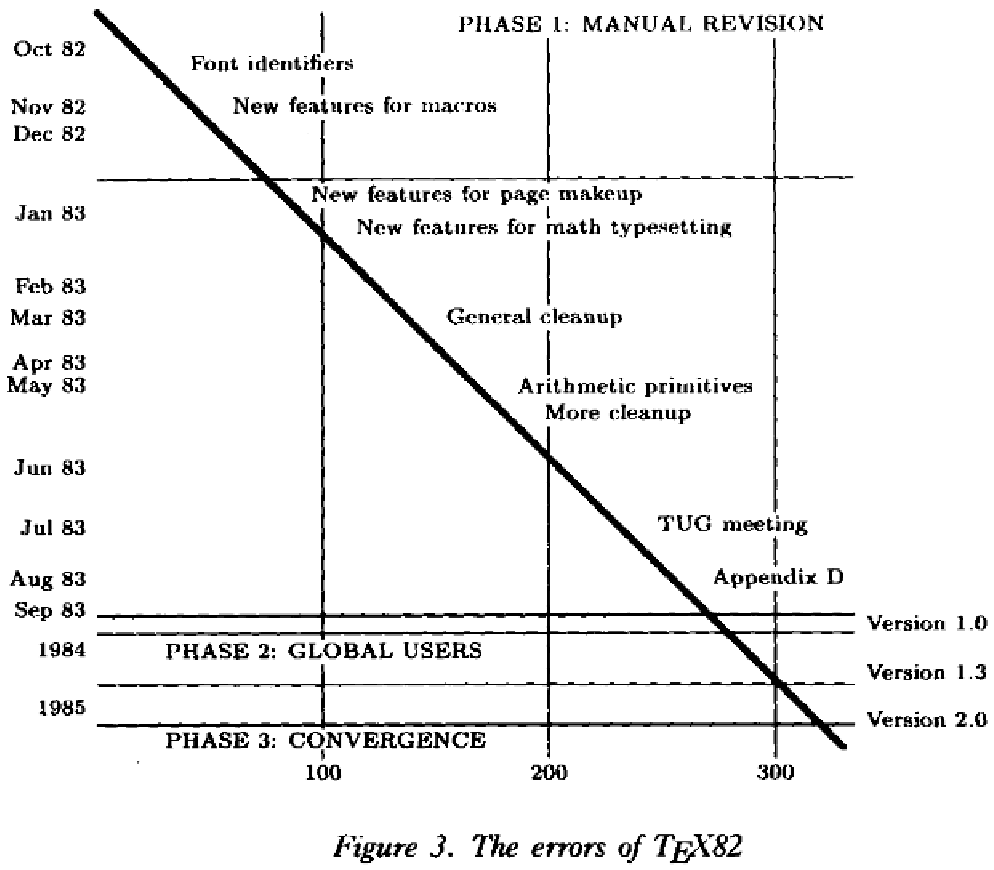


Figure 11.5: Errors of Tex

[See page 18 of Errors of Tex[2].]

Otherwise, the graph would be too crumpled in 1982-1983 area, and/or too sparse in 1984-1985.

11.2.6 Moore's Law

Another well-known graph is Moore's Law Transistor Count: https://en.wikipedia.org/wiki/File:Moore%27s_Law_Transistor_Count_1970-2020.png.

Horizontal — linear scale. Vertical — logarithmic scale.

11.2.7 Money

Banknotes and coins. It's just wouldn't be possible without logarithmic scale.

Here logarithmic scale allows you to pay an arbitrary sum of money using a reasonable number of banknotes/coins.

11.2.8 Sizes, values, weights...

Weights for scales. Same as for money. You can make an arbitrary weight using just a usual set of weights.

Values of resistor's, capacitor's, etc.

Sizes of all nuts, bolts, pipes, etc.

All of them follows logarithmic scales.

[2] https://yurichev.com/mirrors/knuth1989.pdf

11.3 Multiplication and division using addition and subtraction

It is possible to use addition instead of multiplication, using the following rule:

$$\log_{base}(ab) = \log_{base}(a) + \log_{base}(b) \tag{11.3}$$

...while base can be any number.

It's like summing number of zeroes of two numbers. Let's say, you need to multiply 100 by 1000. Just sum number of their zeroes (2 and 3). The result if the number with 5 zeroes. It's the same as $\log_{10}(100)+\log_{10}(1000) = \log_{10}(100000)$.

Division can be replaced with subtraction in the very same way.

11.3.1 Logarithmic slide rule

Here is very typical slide rule[3]. It has many scales, but take a look on C and D scales, they are the same:

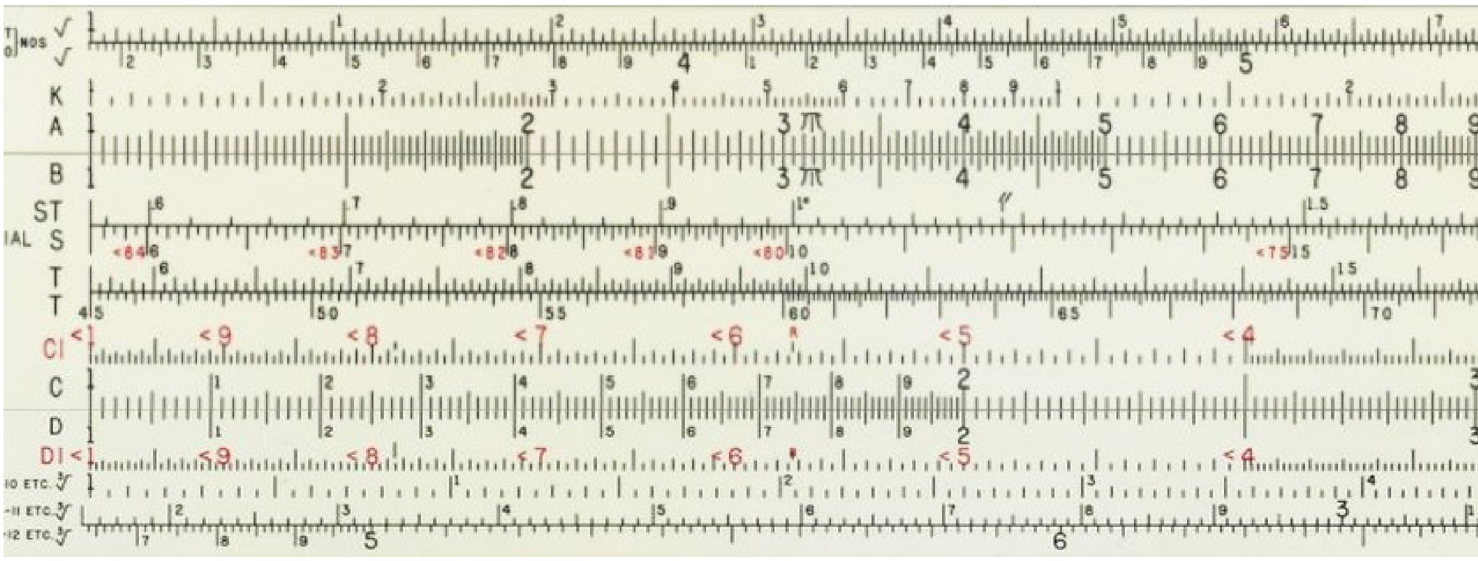


Figure 11.6: Initial state of slide rule

Now shift the core of rule so C scale at 1 will point to 1.2 at D scale:

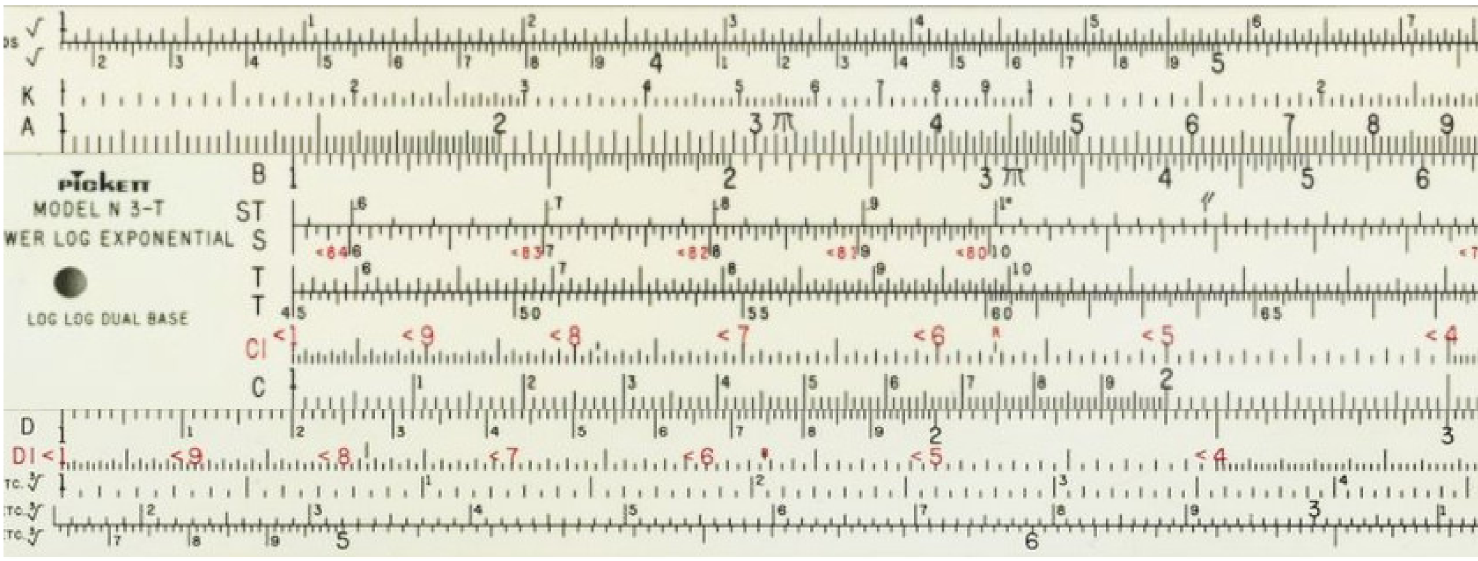


Figure 11.7: C scale shifted

[3]I took screenshots at http://museum.syssrc.com/static/sliderule.html

Find 2 at C scale and find corresponding value at D scale (which is 2.4). Indeed, $1.2 \cdot 2 = 2.4$. It works because by sliding scales we actually add distance between 1 and 1.2 (at any scale) to the distance between 1 and 2 (at any scale). But since these scales logarithmic, addition of logarithmic values is the same as multiplication.

Values on scales can be interpreted as values of other order of magnitude. We can say that 1 at C scale is actually point to 12 at D scale. Find 1.8 at D scale (which is 18 now), it points somewhere between 21 and 22. It's close: $12 \cdot 18 = 216$.

It works because of equation 11.3.

Here is another example from Wikipedia:

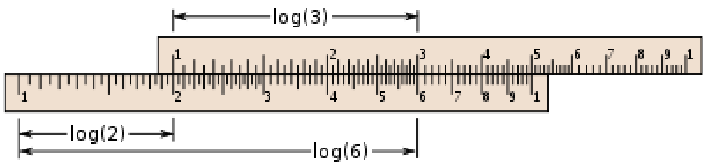


Figure 11.8: Example from Wikipedia

11.3.2 Logarithmic tables

As we can see, the precision of logarithmic slide rule is up to 1 or 2 decimal digits after point. Using precomputed logarithmic tables, it's possible to calculate product of two numbers with a precision up to ≈ 4 digits.

First, find common (base of 10) logarithms of each number using logarithmic table:

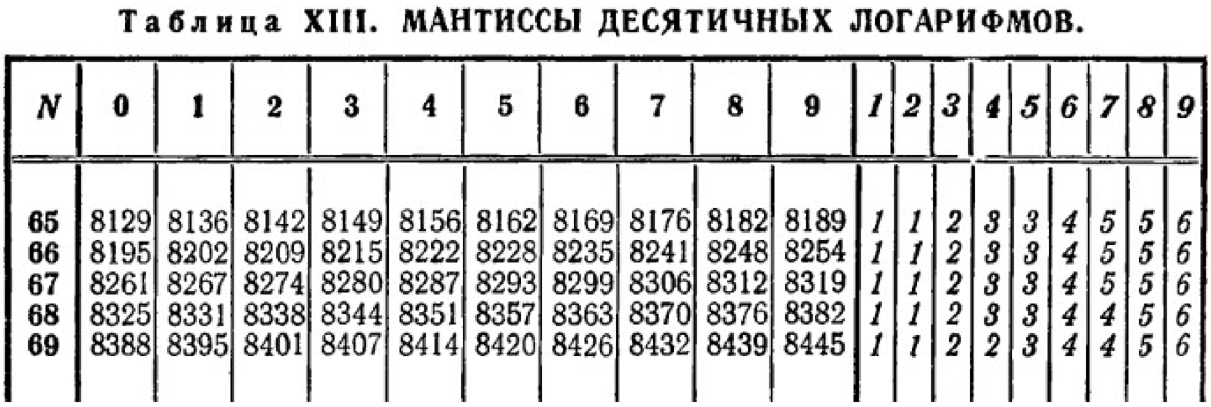


Таблица XIII. МАНТИССЫ ДЕСЯТИЧНЫХ ЛОГАРИФМОВ.

N	0	1	2	3	4	5	6	7	8	9	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6

Figure 11.9: Logarithmic tables

Then add these numbers. Find the number you got in table of powers of 10 (10^x, also called "anti-log table"):

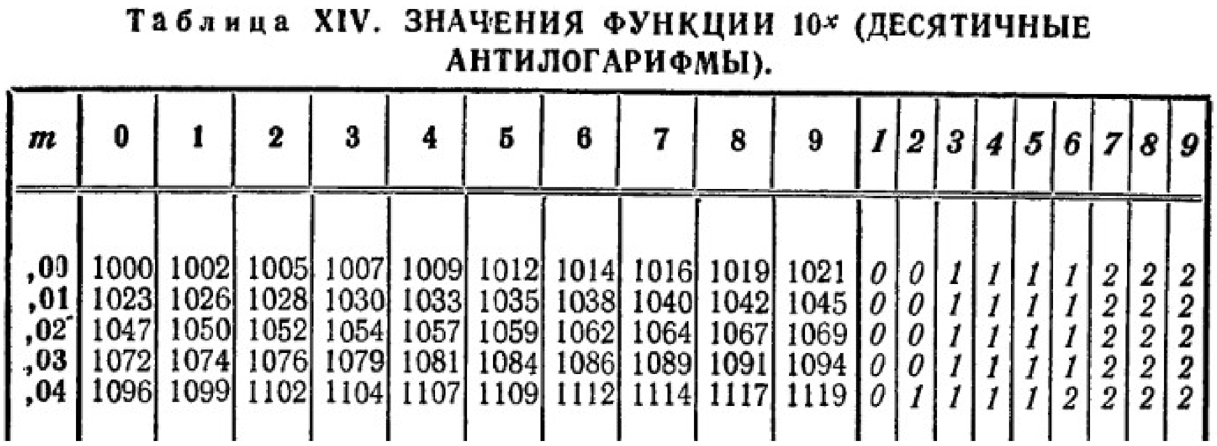


Таблица XIV. ЗНАЧЕНИЯ ФУНКЦИИ 10^x (ДЕСЯТИЧНЫЕ АНТИЛОГАРИФМЫ).

m	0	1	2	3	4	5	6	7	8	9	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
,00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
,01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
,02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
,03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
,04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2

Figure 11.10: Antilog tables

Resulting number is a product. The whole process may be faster than to multiply using long multiplication method using paper-n-pencil taught in schools.

Screenshots I took from the Bradis' book, once popular in USSR. Another well-known book in western world with logarithmic and other tables is Daniel Zwillinger - CRC Standard Mathematical Tables and Formulae (up to 30th edition, the logarithmic tables are dropped after).

11.3.3 Working with very small and very large numbers

It's hard to believe, but the rule used on logarithmic slide rule for multiplication is still used sometimes in software code. It's a problem to work with very small (denormalized) numbers [4] encoded in IEEE 754 standard.

Here is my attempt to calculate $\frac{1.234\times10^{-300}\cdot2.345678901234\times10^{-24}}{3.456789\times10^{-50}}$:

Listing 11.4: C code

```
#include <stdio.h>
#include <math.h>

int main()
{
        double a=1.234e-300;
        double b=2.345678901234e-24;
        double c=3.456789e-50;
        printf ("%.30e\n", a*b/c);
};
```

The output is $1.429261797122261460966983388190 \times 10^{-274}$, which is incorrect. When using debugger, we can see that the multiplication operation raises *inexact exception* and *underflow exception* in FPU. The division operation also raises *inexact exception*.

Let's check in Wolfram Mathematica:

Listing 11.5: Wolfram Mathematica

```
In[]:= a = 1.234*10^(-300);

In[]:= b = 2.345678901234*10^(-24);

In[]:= c = 3.456789*10^(-50);

In[]:= a*b/c
Out[]= 8.37357*10^-275
```

The underflow exception raised in my C program because result of multiplication is in fact $2.894567764122756*10^{-324}$, which is even smaller than smallest denormalized number FPU can work with.

Let's rework our example to compute it all using natural logarithms (`exp(x)` is a C standard function, which computes e^x and `log(x)` here is $\log_e(x)$ (or $ln(x)$)):

Listing 11.6: C code

```
#include <stdio.h>
#include <math.h>

int main()
{
        double a=1.234e-300;
        double b=2.345678901234e-24;
        double c=3.456789e-50;
        printf ("%.30e\n", exp(log(a)+log(b)-log(c)));
};
```

[4] Denormalized numbers in double-precision floating point format are numbers between $\approx 10^{324}$ and $\approx 10^{308}$

Now the output is $8.373573753338710216281125792150 \times 10^{-275}$, same as Mathematica reported.

The same problem with very large numbers.

Listing 11.7: C code

```
#include <stdio.h>
#include <math.h>

int main()
{
        double a=1.234e+300;
        double b=2.345678901234e+24;
        double c=3.456789e+50;
        printf ("%.30e\n", a*b/c);
};
```

When this program running, its result is “inf”, meaning ∞, i.e., overflow occurred. When using debugger, we can see than the multiplication operation raises *inexact exception* plus *overflow exception*. The correct value in Wolfram Mathematica is...

Listing 11.8: Wolfram Mathematica

```
In[]:= a = 1.234*10^300;

In[]:= b = 2.345678901234*10^24;

In[]:= c = 3.456789*10^50;

In[]:= a*b/c
Out[]= 8.37357*10^273
```

Let’s rewrite our C example:

Listing 11.9: C code

```
int main()
{
        double a=1.234e+300;
        double b=2.345678901234e+24;
        double c=3.456789e+50;
        printf ("%.30e\n", exp(log(a)+log(b)-log(c)));
};
```

Now the program reports $8.373573753337712538419923350878 \times 10^{273}$, which is correct value.

The way of representing all numbers as their logarithms called “logarithmic number system” [5]. It allows to work with numbers orders of magnitude lower than FPU can handle.

So why all computations are not performed using logarithms, if it’s so good? It’s better only for very small or very large numbers. Working with small and medium numbers, precision of its logarithmic versions will be much more important and harder to control.

Also, finding logarithm of a number with the following exponentiation are operations may be slower than multiplication itself.

11.3.4 IEEE 754: adding and subtracting exponents

IEEE 754 floating point number consists of sign, significand and exponent. Internally, its simplified representation is:

$$(-1) \cdot sign \cdot significand \times 2^{exponent} \tag{11.4}$$

Given that, the FPU may process significands and exponents separately during multiplication, but when it processes exponents of two numbers, they are just summed up. For example:

[5] https://en.wikipedia.org/wiki/Logarithmic_number_system

$$significand_1 \times 2^{10} \cdot significand_2 \times 2^{50} = significand_3 \times 2^{60} \tag{11.5}$$

...precise values of significands are omitted, but we can be sure, if the first number has exponent of 10, the second has 50, the exponent of the resulting number will be ≈ 60.

Conversely, during division, exponent of divisor is subtracted from the exponent of the dividend.

$$\frac{significand_1 \times 2^{10}}{significand_2 \times 2^{50}} = significand_3 \times 2^{-40} \tag{11.6}$$

I don't have access to Intel or AMD FPU internals, but I can peek into OpenWatcom FPU emulator libraries [6].

Here is summing of exponents during multiplication:
https://github.com/open-watcom/open-watcom-v2/blob/86dbaf24bf7f6a5c270f5a6a50925f468d8d292b/bld/fpuemu/386/asm/fldm386.asm#L212.
And here is subtracting of exponents during division:
https://github.com/open-watcom/open-watcom-v2/blob/e649f6ed488eeebbc7ba9aeed8193d893288d398/bld/fpuemu/386/asm/fldd386.asm#L237.

Here is also multiplication function from FPU emulator in Linux kernel: https://github.com/torvalds/linux/blob/da957e111bb0c189a4a3bf8a00caaecb59ed94ca/arch/x86/math-emu/reg_u_mul.S#L93.

11.4 Exponentiation

Using equation 11.3 we may quickly notice that

$$b^n = \underbrace{b \times \cdots \times b}_{n} = base^{(\log_{base}(b))*n} \tag{11.7}$$

That works with any logarithmic base. In fact, this is the way how exponentiation is computed on computer. x86 CPU and x87 FPU has no special instruction for it.

This is the way how pow() function works in Glibc: https://github.com/lattera/glibc/blob/master/sysdeps/x86_64/fpu/e_powl.S#L189:

Listing 11.10: Glibc source code, fragment of the pow() function

```
...

7:      fyl2x                   // log2(x) : y
8:      fmul    %st(1)          // y*log2(x) : y
        fst     %st(1)          // y*log2(x) : y*log2(x)
        frndint                 // int(y*log2(x)) : y*log2(x)
        fsubr   %st, %st(1)     // int(y*log2(x)) : fract(y*log2(x))
        fxch                    // fract(y*log2(x)) : int(y*log2(x))
        f2xm1                   // 2^fract(y*log2(x))-1 : int(y*log2(x))
        faddl   MO(one)         // 2^fract(y*log2(x)) : int(y*log2(x))
        fscale                  // 2^fract(y*log2(x))*2^int(y*log2(x)) : int(y*log2(x
           ))
        fstp    %st(1)          // 2^fract(y*log2(x))*2^int(y*log2(x))

...
```

x87 FPU has the following instructions used Glibc's version of pow() function: FYL2X (compute $y \cdot log_2 x$), F2XM1 (compute 2^x–1). Even more than that, FYL2X instruction doesn't compute binary logarithm alone, it also performs multiplication operation, to provide more easiness in exponentiation computation.

[6]It was a time in 1980s and 1990s, when FPU was expensive and it could be bought separately in form of additional chip and added to x86 computer. And if you had run a program which uses FPU on the computer where it's missing, FPU emulating library might be an option. Much slower, but better than nothing.

It works because calculating 2^x (exponentiation with base 2) is faster than exponentiation of arbitrary number.

Using hacker's tricks, it's also possible to take advantage of the IEEE 754 format and SSE instructions set: http://stackoverflow.com/a/6486630/4540328.

11.5 Square root

Likewise, square root can be computed in the following way:

$$\sqrt[2]{x} = 2^{\frac{\log_2 x}{2}} \tag{11.8}$$

This leads to an interesting consequence: if you have a value stored in logarithmical form and you need to take square root of it and leave it in logarithmical form, all you need is just to divide it by 2.

And since floating point numbers encoded in IEEE 754 has exponent encoded in logarithmical form, you need just to shift it right by 1 bit to get square root: https://en.wikipedia.org/wiki/Methods_of_computing_square_roots#Approximations_that_depend_on_the_floating_point_representation.

Likewise, cube root and nth root can be calculated using logarithm of corresponding base:

$$\sqrt[b]{x} = b^{\frac{\log_b x}{b}} \tag{11.9}$$

11.6 Base conversion

FYL2X and F2XM1 instructions are the only logarithm-related x87 FPU has. Nevertheless, it's possible to compute logarithm with any other base, using these. The very important property of logarithms is:

$$\log_y(x) = \frac{\log_a(x)}{\log_a(y)} \tag{11.10}$$

So, to compute common (base 10) logarithm using available x87 FPU instructions, we may use this equation:

$$\log_{10}(x) = \frac{\log_2(x)}{\log_2(10)} \tag{11.11}$$

...while $\log_2(10)$ can be precomputed ahead of time.

Perhaps, this is the very reason, why x87 FPU has the following instructions: FLDL2T (load $\log_2(10) = 3.32193...$ constant) and FLDL2E (load $\log_2(e) = 1.4427...$ constant).

Even more than that. Another important property of logarithms is:

$$\log_y(x) = \frac{1}{\log_x(y)} \tag{11.12}$$

Knowing that, and the fact that x87 FPU has FYL2X instruction (compute $y \cdot log_2 x$), logarithm base conversion can be done using multiplication:

$$\log_y(x) = \log_a(x) \cdot \log_y(a) \tag{11.13}$$

So, computing common (base 10) logarithm on x87 FPU is:

$$\log_{10}(x) = \log_2(x) \cdot \log_{10}(2) \tag{11.14}$$

Apparently, that is why x87 FPU has another pair of instructions:

FLDLG2 (load $\log_{10}(2) = 0.30103...$ constant) and FLDLN2 (load $\log_e(2) = 0.693147...$ constant).

Now the task of computing common logarithm can be solved using just two FPU instructions: FYL2X and FLDLG2.

This piece of code I found inside of Windows NT4 (`src/OS/nt4/private/fp32/tran/i386/87tran.asm`), this function is capable of computing both common and natural logarithms:

Listing 11.11: Assembly language code

```
lab fFLOGm
        fldlg2                          ; main LOG10 entry point
        jmp     short fFYL2Xm

lab fFLNm                               ; main LN entry point
        fldln2

lab fFYL2Xm
        fxch
        or      cl, cl                  ; if arg is negative
        JSNZ    Yl2XArgNegative         ; return a NAN
        fyl2x                           ; compute y*log2(x)
        ret
```

11.7 Binary logarithm

Sometimes denoted as lb(), binary logarithms are prominent in computer science, because numbers are usually stored and processed in computer in binary form.

11.7.1 Weights

An interesting problem I've found in the Simon Singh — Fermat's Last Theorem (1997) book:

> The *Arithmetica* which inspired Fermat was a Latin translation made by Claude Gaspar Bachet de Méziriac, reputedly the most learned man in all of France. As well as being a brilliant linguist, poet and classics scholar, Bachet had a passion for mathematical puzzles. His first publication was a compilation of puzzles entitled *Problemes plaisans et délectables qui se font par les nombres*, which included river-crossing problems, a liquid-pouring problem and several think-of-a-number tricks. One of the questions posed was a problem about weights:
>
> > What is the least number of weights that can be used on a set of scales to weigh any whole number of kilograms from 1 to 40
>
> Bachet had a cunning solution which shows that it is possible to achieve this task with only four weights...
>
> ...
>
> In order to weigh any whole number of kilograms from 1 to 40 most people will suggest that six weights are required: 1, 2, 4, 8, 16, 32 kg. In this way, all the weights can easily be achieved by placing the following combinations in one pan:
>
> 1 kg = 1,
> 2 kg = 2,
> 3 kg = 2 + 1,
> 4 kg = 4,
> 5 kg = 4 + 1,
> ...
> 40 kg = 32 + 8.
>
> However, by placing weights in both pans, such that weights are also allowed to sit alongside the object being weighed, Bachet could complete the task with only four weights: 1, 3, 9, 27 kg. A weight placed in the same pan as the object being weighed effectively assumes a negative value. Thus, the weights can be achieved as follows:
>
> 1 kg = 1,
> 2 kg = 3 - 1,

3 kg = 3,
4 kg = 3 + 1,
5 kg = 9 - 3 - 1,
...
40 kg = 27 + 9 + 3 + 1.

We'll talk here only about the first part of the solution: 1/2/4/8/etc kg. weights. Indeed, any weight can be achieved using only a combination of weights in 2^n form.

Say, 11 kg is:

- 1 kg weight = 1 (on)
- 2 kg weight = 1 (on)
- 4 kg weight = 0 (off)
- 8 kg weight = 1 (on)

It's like a number encoded in binary form.

How many weights you would need to weight an n kg? It's $\lceil log_2 n \rceil$.

And $\lceil log_2 40 \rceil = 6$.

As of the second part of the problem, "SAT/SMT by Example"[7]has an example about it.

11.7.2 A burned LED garland

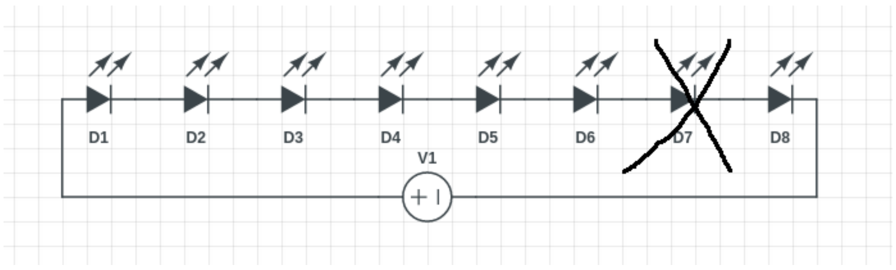


Figure 11.11: A burned LED garland

It doesn't work: LED D7 has been burned, but yet, you don't know, which. You have an ohmmeter to find it. You can check them all, but there 8 of them. Or, you can divide that chain by two and try the left pin of D1 and the right pin of D4. Working? If yes, try the left pin of D4 and the right pin of D8. It doesn't. No you have a failed chain of 4 LEDs, you know that the right half of chain is failing, but the left is OK. You divide that chain by two and repeat the process. Approximately, a number of all measurements is a binary logarithm of number of LEDs.

At best (by luck, you always hit failed part of chain), a number of all measurements is binary logarithm of number of LEDs (8) minus 1, and that is 3. At worst (being unlucky, you always hit working part of chain then you switch to failed part), is the same number, doubled, that is 6.

11.7.3 Browing a telephone book

Another example:

In daily life a telephone book can be used as a one-way function; given a name one can easily find the corresponding telephone number but not the other way around. Looking up a telephone number of a person amounts to finding the name of that person. This takes log_2 L operations, if L is the number of names in the telephone guide. Finding the name if the telephone number is given means going through the whole book, name after name. The complexity is L. Property F2 is based on the exponential relation between log_2 L and L.

[7]https://sat-smt.codes/

11.7.4 Denoting a number of bits for some value

How many bits we need to allocate to store googol number (10^{100})?

Listing 11.12: Wolfram Mathematica

```
In[]:= Log2[10^100] // N
Out[]= 332.193
```

Binary logarithm of some number is the number of how many bits needs to be allocated.

If you have a variable which always has 2^x form, it's a good idea to store a binary logarithmic representation ($\log_2(x)$) instead of it. There are at least two reasons: 1) the programmer shows to everyone that the number has always 2^x form; 2) it's error-prone, it's not possible to accidentally store a number in some other form to this variable, so this is some kind of protection; 3) logarithmic representation is more compact. There is, however, performance issue: the number must be converted back, but this is just one shifting operation (`1<<log_n`).

Here is an example from NetBSD NTP client (`netbsd-5.1.2/usr/src/dist/ntp/include/ntp.h`):

Listing 11.13: C code

```
/*
 * Poll interval parameters
 */
...
#define NTP_MINPOLL     4       /* log2 min poll interval (16 s) */
#define NTP_MINDPOLL    6       /* log2 default min poll (64 s) */
#define NTP_MAXDPOLL    10      /* log2 default max poll ( 17 m) */
#define NTP_MAXPOLL     17      /* log2 max poll interval ( 36 h) */
```

Couple examples from zlib (`deflate.h`):

Listing 11.14: C code

```
    uInt  w_size;        /* LZ77 window size (32K by default) */
    uInt  w_bits;        /* log2(w_size)  (8..16) */
    uInt  w_mask;        /* w_size - 1 */
```

Another piece from zlib (`contrib/blast/blast.c`):

Listing 11.15: C code

```
    int dict;           /* log2(dictionary size) - 6 */
```

If you need to generate bitmasks in range 1, 2, 4, 8...0x80000000, it is good idea to assign self-documenting name to iterator variable:

Listing 11.16: C code

```
for (log2_n=1; log2_n<32; log2_n++)
    1<<log2_n;
```

Now about compactness, here is the fragment I found in OpenBSD, related to SGI IP22 architecture [8] (`OS/OpenBSD/sys/arch/sgi/sgi/ip22_machdep.c`):

Listing 11.17: C code

```
                 /*
                  * Secondary cache information is encoded as WWLLSSSS, where
                  * WW is the number of ways
                  *   (should be 01)
                  * LL is Log2(line size)
                  *   (should be 04 or 05 for IP20/IP22/IP24, 07 for IP26)
```

[8] http://www.linux-mips.org/wiki/IP22

```
                 * SS is Log2(cache size in 4KB units)
                 *   (should be between 0007 and 0009)
                 */
```

Here is another example of using binary logarithm in Mozilla JavaScript engine (JIT compiler) [9]. If some number is multiplied by 2^x, the whole operation can be replaced by bit shift left.
The following code (`js/src/jit/mips/CodeGenerator-mips.cpp`), when translating multiplication operation into MIPS machine code, first, get assured if the number is really has 2^x form, then it takes binary logarithm of it and generates MIPS SLL instruction, which states for "Shift Left Logical".

Listing 11.18: Mozilla JavaScript JIT compiler (translating multiplication operation into MIPS bit shift instruction)

```
bool
 CodeGeneratorMIPS::visitMulI(LMulI *ins)
 {
           default:
             uint32_t shift = FloorLog2(constant);

             if (!mul->canOverflow() && (constant > 0)) {
                 // If it cannot overflow, we can do lots of optimizations.
                 uint32_t rest = constant - (1 << shift);

                 // See if the constant has one bit set, meaning it can be
                 // encoded as a bitshift.
                 if ((1 << shift) == constant) {
                     masm.ma_sll(dest, src, Imm32(shift));
                     return true;
                 }

...
```

Thus, for example, $x = y \cdot 1024$ (which is the same as $x = y \cdot 2^{10}$) translates into $x = y << 10$.

11.7.5 Calculating binary logarithm

If all you need is integer result of binary logarithm ($abs(\log_2(x))$ or $\lfloor \log_2(x) \rfloor$), calculating is just counting all binary digits in the number minus 1. In practice, this is the task of calculating leading zeroes.

Here is example from Mozilla libraries (`mfbt/MathAlgorithms.h` [10]):

Listing 11.19: Mozilla libraries

```
class FloorLog2<T, 4>
{
public:
   static uint_fast8_t compute(const T aValue)
   {
     return 31u - CountLeadingZeroes32(aValue | 1);
   }
 };

inline uint_fast8_t
 CountLeadingZeroes32(uint32_t aValue)
 {
   return __builtin_clz(aValue);
 }
```

Latest x86 CPUs has LZCNT (Leading Zeroes CouNT) instruction for that [11], but there is also BSR (Bit Scan Reverse) instruction appeared in 80386, which can be used for the same purpose. More information about this instruction on various architectures: https://en.wikipedia.org/wiki/Find_first_set.

[9] http://fossies.org/linux/seamonkey/mozilla/js/src/jit/mips/CodeGenerator-mips.cpp
[10] http://fossies.org/linux/seamonkey/mozilla/mfbt/MathAlgorithms.h
[11] GNU `__builtin_clz()` function on x86 architecture can be thunk for LZCNT

There are also quite esoteric methods to count leading zeroes without this specialized instruction: http://yurichev.com/blog/de_bruijn/.

11.7.6 O(log n) time complexity

Time complexity[12] is a measure of speed of a specific algorithm in relation to the size of input data.

O(1) – time is always constant, to matter what size of input data. Simplest example is object getter – it just returns some value.

O(n) – time is linear, growing according to the size of input data. Simplest example is search for some value in the input array. The larger array, the slowest search.

O(log n) – time is logarithmic to the input data. Let's see how this can be.

Let's recall child's number guessing game[13]. One player think about some number, the other should guess it, offering various versions. First player answers, is guessed number is larger or less. A typical dialogue:

```
-- I think of a number in 1..100 range.
-- Is it 50?
-- My number is larger.
-- 75?
-- It is lesser.
-- 63?
-- Larger.
-- 69?
-- Larger.
-- 72?
-- Lesser.
-- 71?
-- Correct.
```

Best possible strategy is to divide the range in halves. The range is shorten at each step by half. At the very end, the range has length of 1, and this is correct answer. Maximal number of steps using the strategy described here are $\log_2(initial\_range)$. In our example, initial range is 100, so the maximum number of steps is 6.64... or just 7. If the initial range is 200, maximum number of steps are $\log_2(200) = 7.6..$ or just 8. The number of steps increasing by 1 when the range is doubled. Indeed, doubled range indicates that the guesser needs just one more step at the start, not more. If the initial range is 1000, numbers of steps are $\log_2(1000) = 9.96...$ or just 10.

This is exactly O(log n) time complexity.

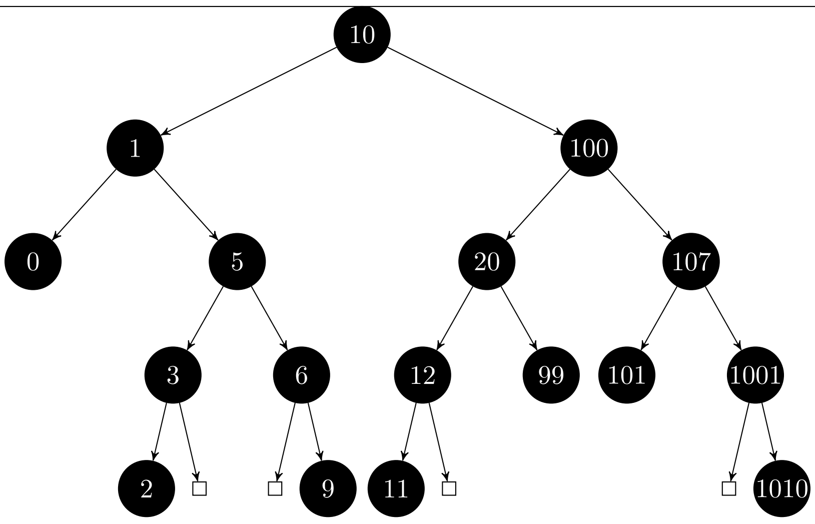
Now let's consider couple of practical real-world algorithms. One interesting thing is that if the input array is sorted, and its size is known, and we need to find some value in it, the algorithm works exactly in the same way as child's number guessing game! The algorithm starts in the middle of array and compare the value there with the value sought-after. Depending on the result (larger or lesser), the *cursor* is moved left or right and operating range is decreasing by half. This is called binary search[14], and there is the `bsearch()` function in standard C/C++ library[15].

Here is how binary search is used in git: https://www.kernel.org/pub/software/scm/git/docs/git-bisect.html.

Another prominent example in CS is binary trees. They are heavily used internally in almost any programming language, when you use set, map, dictionary, etc.

Here is a simple example with the following numbers (or keys) inserted into binary tree: 0, 1, 2, 3, 5, 6, 9, 10, 11, 12, 20, 99, 100, 101, 107, 1001, 1010.

[12]https://en.wikipedia.org/wiki/Time_complexity
[13]http://rosettacode.org/wiki/Guess_the_number
[14]https://en.wikipedia.org/wiki/Binary_search_algorithm
[15]http://en.cppreference.com/w/cpp/algorithm/bsearch



And here is how binary tree search works: put *cursor* at the root. Now compare the value under it with the value sought-after. If the value we are seeking for is lesser than the current, take a move into left node. If it's bigger, move to the right node. Hence, each left descendant node has value lesser than in ascendant node. Each right node has value which is bigger. The tree must be rebalanced after each modification (I gave examples of it in my book about reverse engineering (`http://beginners.re/`, 51.4.4)). Nevertheless, lookup function is very simple, and maximal number of steps is $\log_n(number\_of\_nodes)$. We've got 17 elements in the tree at the picture, $\log_2(17) = 4.08...$, indeed, there are 5 tiers in the tree.

11.8 Common (base 10) logarithms

Also known as "decimal logarithms". Denoted as `lg` on handheld calculators.

10 is a number inherently linked with human's culture, since almost all humans has 10 digits. Decimal system is a result of it. Nevertheless, 10 has no special meaning in mathematics and science in general. So are common logarithms.

One notable use is a decibel logarithmic scale, which is based of common logarithm.

11.9 Printing

Common logarithms are sometimes used to calculate space for decimal number in the string or on the screen. How many characters you should allocate for 64-bit number? 20, because $log_{10}(2^{64}) = 19.2...$.

Functions like `itoa()`[16] (which converts input number to a string) can calculate output buffer size precisely, calculating common logarithm of the input number.

11.9.1 Hexdump and binary logarithm

This is one down-to-programmer's-earth application of binary algorithms.

I wanted to print a hexdump of a buffer. Each line is preceded by an address:

```
0x0 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x10 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
0xe0 e0 e1 e2 e3 e4 e5 e6 e7-e8 e9 ea eb ec ed ee ef "................"
...
0xf0 f0 f1 f2 f3 f4 f5 f6 f7-f8 f9 fa fb fc fd fe ff "................"
0x100 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x110 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
```

No good. But we can print addresses as 32-bit values:

```
0x00000000 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x00000010 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
0x000000e0 e0 e1 e2 e3 e4 e5 e6 e7-e8 e9 ea eb ec ed ee ef "................"
```

[16] `http://www.cplusplus.com/reference/cstdlib/itoa/`

```
...
0x000000f0 f0 f1 f2 f3 f4 f5 f6 f7-f8 f9 fa fb fc fd fe ff "................"
0x00000100 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x00000110 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
```

Fine, but what would you do if you have 64-bit addresses? This is too bulky:

```
0x0000000000000000 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x0000000000000010 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
0x00000000000000e0 e0 e1 e2 e3 e4 e5 e6 e7-e8 e9 ea eb ec ed ee ef "................"
...
0x00000000000000f0 f0 f1 f2 f3 f4 f5 f6 f7-f8 f9 fa fb fc fd fe ff "................"
0x0000000000000100 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x0000000000000110 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
```

One solution is to align all values somehow according to the last value (or address). How many hex digits we need to print the last (0x110) address? 3. So let's all addresses will be printed as 3-digit numbers.

To get exact number of hex digits, we just use binary logarithm.

$\lceil \frac{log_2(0x110)}{4} \rceil$

(Hexadecimal logarithm would also suffice.)

Why ceiling? Say, the last address has 11 bits, and 11/4=2, but to print 11 binary digits, we want 3 hexadecimal digits. So the result of division must be "ceiled" to 3.

Now it's neat:

```
0x000 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x010 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
0x0e0 e0 e1 e2 e3 e4 e5 e6 e7-e8 e9 ea eb ec ed ee ef "................"
...
0x0f0 f0 f1 f2 f3 f4 f5 f6 f7-f8 f9 fa fb fc fd fe ff "................"
0x100 00 01 02 03 04 05 06 07-08 09 0a 0b 0c 0d 0e 0f "................"
0x110 10 11 12 13 14 15 16 17-18 19 1a 1b 1c 1d 1e 1f "................"
```

The code:

```
...
// http://stackoverflow.com/questions/11376288/fast-computing-of-log2-for-64-bit-integers
static const int tab64[64]=
{
        63,  0, 58,  1, 59, 47, 53,  2,
        60, 39, 48, 27, 54, 33, 42,  3,
        61, 51, 37, 40, 49, 18, 28, 20,
        55, 30, 34, 11, 43, 14, 22,  4,
        62, 57, 46, 52, 38, 26, 32, 41,
        50, 36, 17, 19, 29, 10, 13, 21,
        56, 45, 25, 31, 35, 16,  9, 12,
        44, 24, 15,  8, 23,  7,  6,  5
};

uint64_t uint64_log2 (uint64_t value)
{
        value |= value >> 1;
        value |= value >> 2;
        value |= value >> 4;
        value |= value >> 8;
        value |= value >> 16;
        value |= value >> 32;
        return tab64[((uint64_t)((value - (value >> 1))*0x07EDD5E59A4E28C2)) >> 58];
}

```

```
unsigned division_ceil (unsigned x, unsigned y)
{
        //
           https://stackoverflow.com/questions/2745074/fast-ceiling-of-an-integer-division-in-c-c
        return (x + y - 1) / y;
};

void Log::hexdump(BYTE *buf, size_t size, size_t ofs)
{
        size_t last_address_to_be_printed=ofs+size;
        unsigned fill_zeroes;
        if (last_address_to_be_printed==0)
        {
                fill_zeroes=0;
        }
        else
        {
                unsigned binary_digits=uint64_log2(last_address_to_be_printed)+1;
                fill_zeroes=division_ceil(binary_digits, 4);
        };

...

                out << "0x" << std::hex << std::setw(fill_zeroes) << (starting_offset
                    + pos + ofs) << " ";
...
```

Now the code of my small logger library: https://αβγ.ελ/current_tree/log/logger_code

11.9.2 Naive approach

I've found this is in a real code:

```
ostream & operator << (ostream & os, const FmtNat & fn) {
  int64_t n = fn.num;
  assert (n >= 0);                                          // 123456
  if (n < 10)                    os << "      " << n;       // .....n
  else if (n < 100)              os << "     " << n;        // ....nn
  else if (n < 1000)             os << "    " << n;         // ...nnn
  else if (n < 10000)            os << "   " << n;          // ..nnnn
  else if (n < 100000)           os << "  " << n;           // .nnnnn
  else if (n < 1000000)          os << n;                   // nnnnnn
  else if (n < 10000000)         os << n/1000         << "e3"; // nnnne3
  else if (n < 100000000)        os << n/10000        << "e4"; // nnnne4
  else if (n < 1000000000)       os << n/100000       << "e5"; // nnnne5
...
```

This is like reinvention of decimal/common logarithm.

11.10 Natural logarithm

Natural logarithm (denoted as `ln` on handheld calculators, and sometimes denoted just as `log`) is logarithm of base $e = 2.718281828...$. Where this constant came from?

11.10.1 Savings account in your bank

Let's say you make a deposit into bank, say, 100 dollars (or any other currency). They offer 2.5% per year (annual percentage yield). This mean, you'll can get doubled amount of money (200 dollars) after 40 years. So far so good. But some banks offers compound interest. Also called "complex percent" in Russian language, where "complex" in this phrase is closer to the word "folded". This mean, after each year, they pretend you withdraw your money with interest, then redeposit them instantly. Banks also say that the interest is recapitalized once a year. Let's calculate final amount of money after 40 years:

Listing 11.20: Python code

```
#!/usr/bin/env python

initial=100 # 100 dollars, or any other currency
APY=0.025   # Annual percentage yield = 2.5%

current=initial

# 40 years
for year in range(40):
    # what you get at the end of each year?
    current=current+current*APY
    print "year=", year, "amount at the end", current
```

```
year= 0 amount at the end 102.5
year= 1 amount at the end 105.0625
year= 2 amount at the end 107.6890625
year= 3 amount at the end 110.381289063
...
year= 36 amount at the end 249.334869861
year= 37 amount at the end 255.568241608
year= 38 amount at the end 261.957447648
year= 39 amount at the end 268.506383839
```

The thing is that the final amount (268.50...) is aimed toward e constant.

Now there is another bank, which offers to recapitalize your deposit each month. We'll rewrite our script slightly:

Listing 11.21: Python code

```
#!/usr/bin/env python

initial=100 # $100
APY=0.025 # Annual percentage yield = 2.5%

current=initial

# 40 years
for year in range(40):
    for month in range(12):
        # what you get at the end of each month?
        current=current+current*(APY/12)
        print "year=", year, "month=", month, "amount", current
```

```
year= 0 month= 0 amount 100.208333333
year= 0 month= 1 amount 100.417100694
year= 0 month= 2 amount 100.626302988
year= 0 month= 3 amount 100.835941119
...
year= 39 month= 8 amount 269.855455383
year= 39 month= 9 amount 270.417654248
year= 39 month= 10 amount 270.981024361
year= 39 month= 11 amount 271.545568162
```

The final result is even closer to e constant.

Let's imagine there is a bank which allows to recapitalize each day:

Listing 11.22: Python code

```
#!/usr/bin/env python

initial=100 # $100
APY=0.025 # Annual percentage yield = 2.5%

current=initial

# 40 years
for year in range(40):
    for month in range(12):
        for day in range(30):
            # what you get at the end of each day?
            current=current+current*(APY/12/30)
            print "year=", year, "month=", month, "day=", day, "amount", current
```

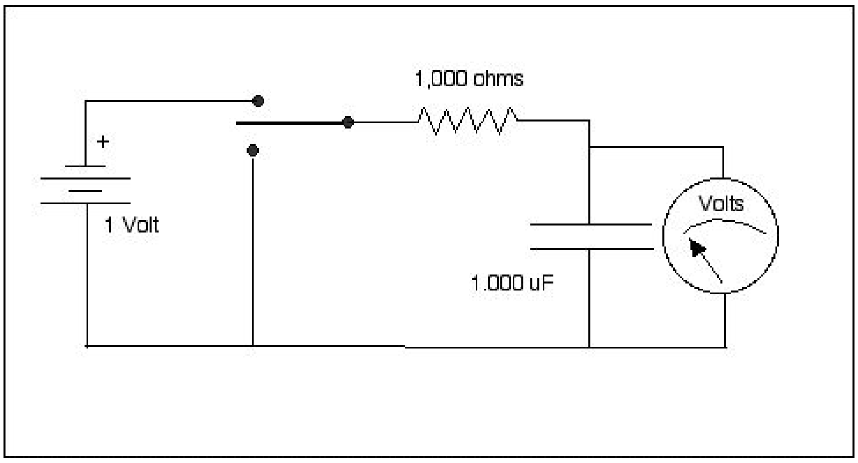
```
year= 0 month= 0 day= 0 amount 100.006944444
year= 0 month= 0 day= 1 amount 100.013889371
year= 0 month= 0 day= 2 amount 100.02083478
year= 0 month= 0 day= 3 amount 100.027780671
...
year= 39 month= 11 day= 26 amount 271.762123927
year= 39 month= 11 day= 27 amount 271.780996297
year= 39 month= 11 day= 28 amount 271.799869977
year= 39 month= 11 day= 29 amount 271.818744968
```

The final amount of money is more closer to e constant.

If to imagine some really crazy bank client who redeposit his deposit infinite number of times per each day, the final value after 40 years would be $100 \cdot e$. It's not possible in the real world, so the final amount is approaches this value, but is never equal to it. Mathematically speaking, its limit is $100 \cdot e$.

11.10.2 Exponential decay

Capacitor discharge



From electronics engineering course we may know that the capacitor discharging by half after $RC\ln(2)$ seconds, where C is capacity of capacitor in farads and R resistance of resistor in ohms. Given $1k\Omega$ resistor and $1000\mu F$ capacitor, what its voltage after 1 seconds will be? after 2 seconds? It's discharge can be calculated using this equation:

$$V = V_0 \cdot e^{\frac{-t}{RC}}$$

...where V_0 is initial charge in volts, t is time in seconds and e is base of natural logarithm.

Let's see it in Wolfram Mathematica:

Listing 11.23: Wolfram Mathematica

```
r = 1000; (* resistance in ohms *)

c = 0.001; (* capacity in farads *)

v = 1; (* initial voltage *)

Plot[v*E^((-t)/(r*c)), {t, 0, 5},
 GridLines -> {{Log[2], Log[2]*2, Log[2]*3}, {0.5, 0.25, 0.125}},
 Epilog -> {Text["ln(2)", {Log[2], 0.05}],
   Text["ln(2)*2", {Log[2]*2, 0.05}],
   Text["ln(2)*3", {Log[2]*3, 0.05}],
   Text["1/2", {0.1, 0.5}], Text["1/4", {0.1, 0.25}],
   Text["1/8", {0.1, 0.128}]}, AxesLabel -> {seconds, voltage}]
```

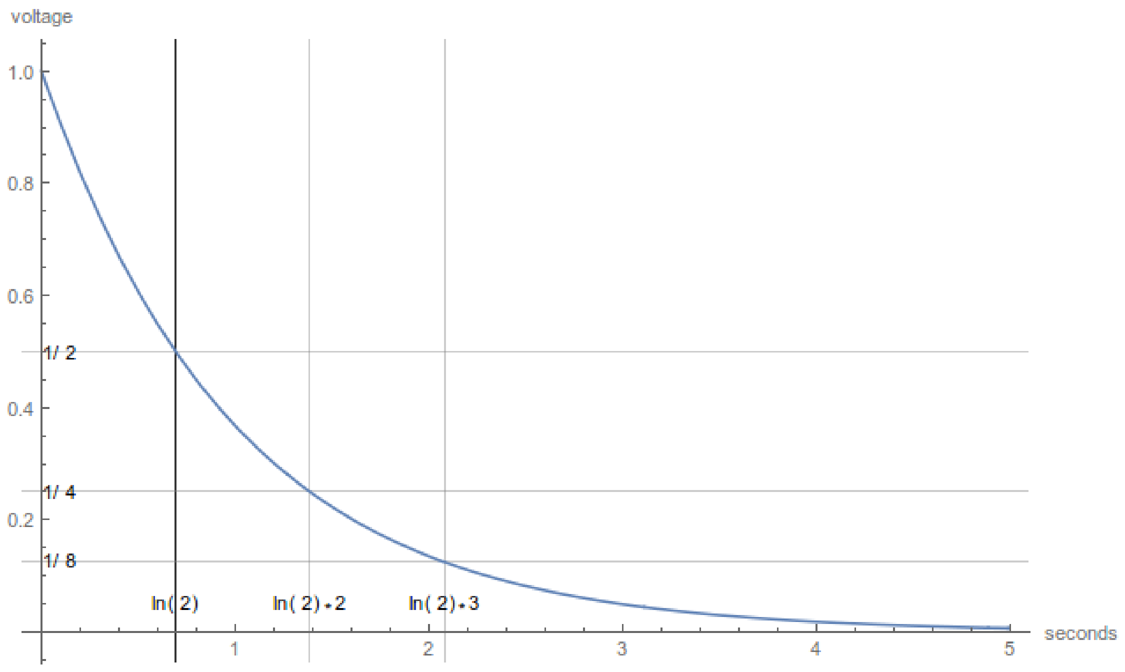


Figure 11.12: Capacitor voltage during discharge

As we can see, $\frac{1}{2}$ of initial charge is left after $ln(2)$ seconds (0.69...), and $\frac{1}{4}$ of charge is left after $ln(4)$ seconds (1.38...). Indeed, if we interesting in precise time in seconds, when charge will be $\frac{1}{x}$, just calculate $ln(x)$.

Now here is the same plot, but I added two more labels, $\frac{1}{3}$ and $\frac{1}{7}$:

Listing 11.24: Wolfram Mathematica

```
Plot[v*E^((-t)/(r*c)), {t, 0, 5},
 GridLines -> {{Log[3], Log[7]}, {1/3, 1/7}},
 Epilog -> {Text["ln(3)", {Log[3], 0.05}],
   Text["ln(7)", {Log[7], 0.05}],
   Text["1/3", {0.1, 1/3}], Text["1/7", {0.1, 1/7}]},
 AxesLabel -> {seconds, voltage}]
```

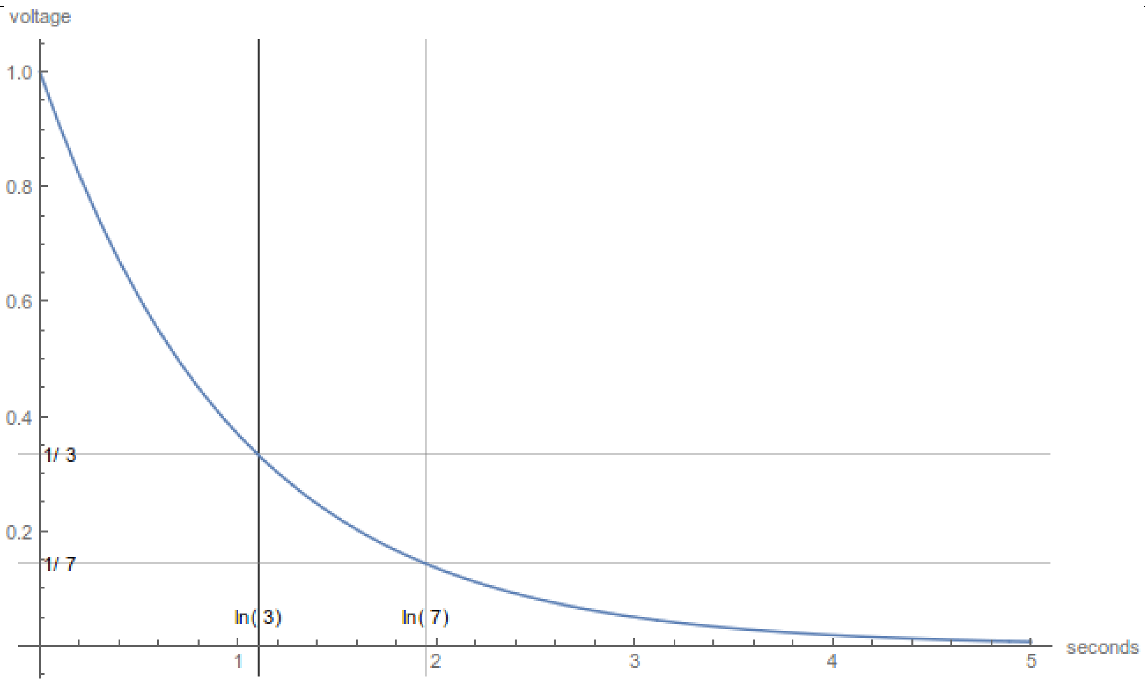


Figure 11.13: Capacitor voltage during discharge

...and we see that these points corresponds to $ln(3)$ and $ln(7)$. That means, $\frac{1}{3}$ of charge is left after $ln(3) \approx 1.098...$ seconds and $\frac{1}{7}$ of charge after $ln(7) \approx 1.945...$ seconds.

Radioactive decay

Radioactive decay is also exponential decay. Let's take Polonium 210 as an example[17]. It's half-life (calculated) is ≈ 138.376 days. That means that if you've got 1kg of Polonium 210, after ≈ 138 days, half of it (0.5 kg) left as ^{210}Po and another half is transformed into ^{206}Pb (isotope of lead[18]). After another ≈ 138 days, you'll get $\frac{3}{4}$ of isotope of lead and $\frac{1}{4}$ will left as ^{210}Po. After another ≈ 138 days, amount of Polonium will be halved yet another time, etc.

The equation of radioactive decay is:

$$N = N_0 e^{-\lambda t}$$

...where N is number of atoms at some point of time, N_0 is initial number of atoms, t is time, λ is decay constant. Decay of Polonium is exponential, but decay constant is the constant, defining how fast (or slow) it will fall.

Here we go in Mathematica, let's get a plot for 1000 days:

Listing 11.25: Wolfram Mathematica

```
l = 0.005009157516910051; (* decay constant of Polonium 210 *)

hl = Log[2]/l
138.376

Plot[E^(-l*t), {t, 0, 1000},
 GridLines -> {{hl, hl*2, hl*3}, {0.5, 0.25, 0.125}},
 Epilog -> {Text["hl", {hl, 0.05}], Text["hl*2", {hl*2, 0.05}],
   Text["hl*3", {hl*3, 0.05}], Text["1/2", {30, 0.5}],
   Text["1/4", {30, 0.25}], Text["1/8", {30, 0.128}]},
 AxesLabel -> {days, atoms}]
```

[17] https://en.wikipedia.org/wiki/Polonium
[18] https://en.wikipedia.org/wiki/Isotopes_of_lead#Lead-206

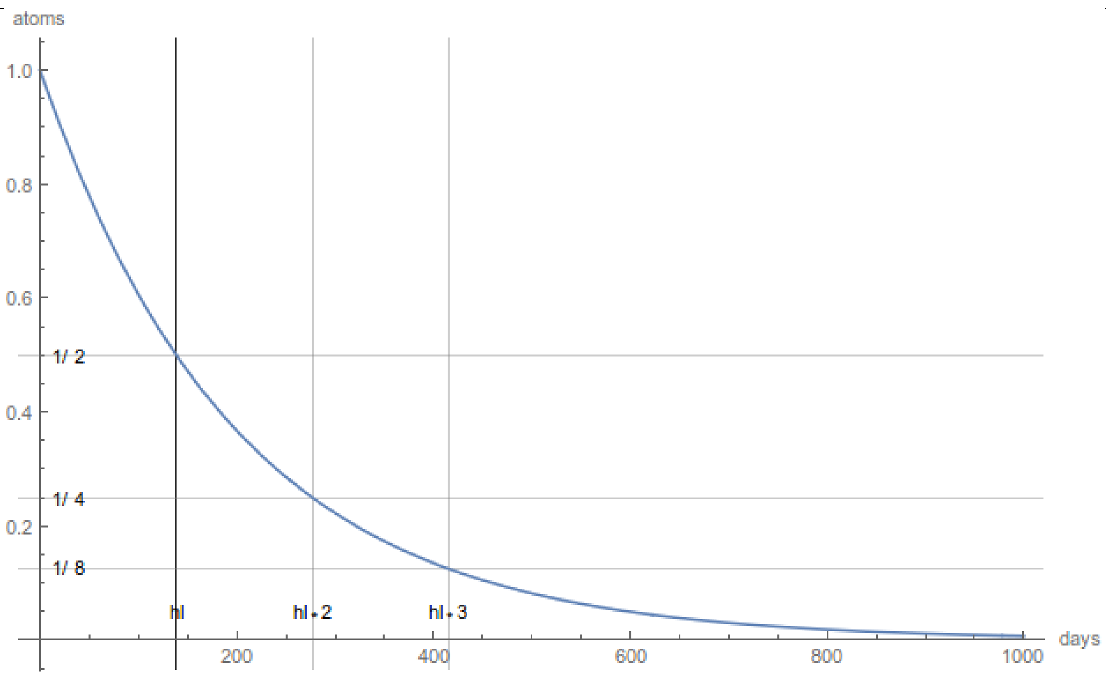


Figure 11.14: Exponential decay of Polonium 210

Beer froth

There is even the paper (got Ig Nobel prize in 2002), author's of which demonstrates that beer froth is also decays exponentially: http://iopscience.iop.org/0143-0807/23/1/304/, https://classes.soe.ucsc.edu/math011a/Winter07/lecturenotes/beerdecay.pdf.

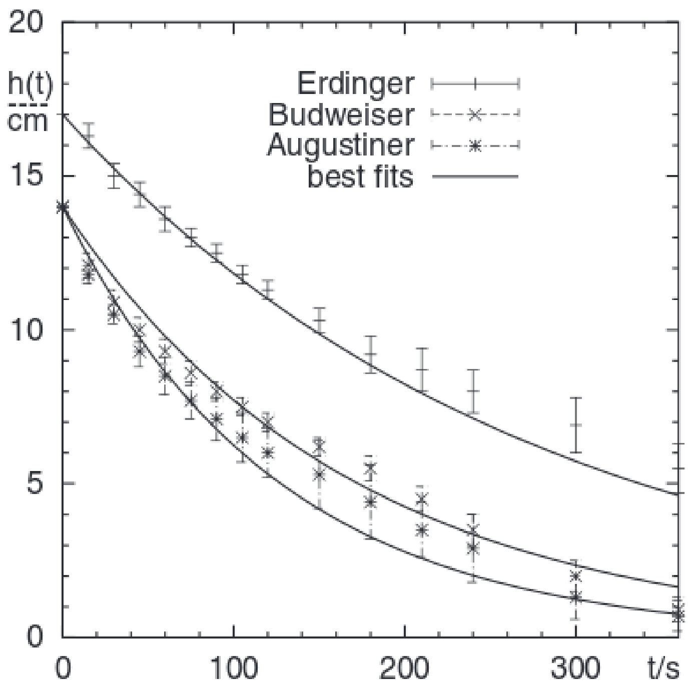


Figure 11.15: Results from the paper

The paper can be taken as a joke, nevertheless, it's a good demonstration of exponential decay.

Conclusion

Capacitor discharge and radioactive decay obeys the same law of halving some amount after equal gaps of time:

$$amount = amount_0 \cdot e^{-decay\_constant \cdot time}$$

Decay constant in case of capacitor discharge defined by product of resistance and capacity. The bigger one of them, the slower decay.

Natural logarithm is used to calculate gap of time (half-life or half-time) judging by decay constant.

11.11 Negative logarithm

By convention, negative logarithm is...

Listing 11.26: Wolfram Mathematica

```
In[1]:= Log[2,1/2]
Out[1]= -1
In[2]:= Log[2,1/16]
Out[2]= -4
```

That corresponds to...

Listing 11.27: Wolfram Mathematica

```
In[3]:= 2^(-4)
Out[3]= 1/16
```

Chapter 12

Symbolic computation

We will then augment the representational power of our language by introducing symbolic expressions — data whose elementary parts can be arbitrary symbols rather than only numbers.

Structure and Interpretation of Computer Programs

Some numbers can only be represented in binary system approximately, like $\frac{1}{3}$ and π. If we calculate $\frac{1}{3} \cdot 3$ step-by-step, we may have loss of significance. We also know that $sin(\frac{\pi}{2}) = 1$, but calculating this expression in usual way, we can also have some noise in result. Arbitrary-precision arithmetic[1] is not a solution, because these numbers cannot be stored in memory as a binary number of finite length.

How we could tackle this problem? Humans reduce such expressions using paper and pencil without any calculations. We can mimic human behaviour programmatically if we will store expression as tree and symbols like π will be converted into number at the very last step(s).

This is what Wolfram Mathematica[2] does. Let's start it and try this:

```
In[]:= x + 2*8
Out[]= 16 + x
```

Since Mathematica has no clue what x is, it's left *as is*, but $2 \cdot 8$ can be reduced easily, both by Mathematica and by humans, so that is what has done. In some point of time in future, Mathematica's user may assign some number to x and then, Mathematica will reduce the expression even further.

Mathematica does this because it parses the expression and finds some known patterns. This is also called *term rewriting*[3]. In plain English language it may sounds like this: "if there is a + operator between two known numbers, replace this subexpression by a computed number which is sum of these two numbers, if possible". Just like humans do.

Mathematica also has rules like "replace $sin(\pi)$ by 0" and "replace $sin(\frac{\pi}{2})$ by 1", but as you can see, π must be preserved as some kind of symbol instead of a number.

So Mathematica left x as unknown value. This is, in fact, common mistake by Mathematica's users: a small typo in an input expression may lead to a huge irreducible expression with the typo left.

Another example: Mathematica left this deliberately while computing binary logarithm:

```
In[]:= Log[2, 36]
Out[]= Log[36]/Log[2]
```

[1] https://en.wikipedia.org/wiki/Arbitrary-precision_arithmetic
[2] Another well-known symbolic computation system are Maxima and SymPy
[3] https://en.wikipedia.org/wiki/Rewriting

Because it has a hope that at some point in future, this expression will become a subexpression in another expression and it will be reduced nicely at the very end. But if we really need a numerical answer, we can force Mathematica to calculate it:

```
In[]:= Log[2, 36] // N
Out[]= 5.16993
```

Sometimes unresolved values are desirable:

```
In[]:= Union[{a, b, a, c}, {d, a, e, b}, {c, a}]
Out[]= {a, b, c, d, e}
```

Characters in the expression are just unresolved symbols[4] with no connections to numbers or other expressions, so Mathematica left them *as is*.

Another real world example is symbolic integration[5], i.e., finding formula for integral by rewriting initial expression using some predefined rules. Mathematica also does it:

```
In[]:= Integrate[1/(x^5), x]
Out[]= -(1/(4 x^4))
```

Benefits of symbolic computation are obvious: it is not prone to loss of significance[6] and round-off errors[7], but drawbacks are also obvious: you need to store expression in (possible huge) tree and process it many times. Term rewriting is also slow. All these things are extremely clumsy in comparison to a fast FPU[8].

“Symbolic computation” is opposed to “numerical computation”, the last one is just processing numbers step-by-step, using calculator, CPU[9] or FPU.

Some task can be solved better by the first method, some others – by the second one.

12.1 Rational data type

Some LISP implementations can store a number as a ratio/fraction [10], i.e., placing two numbers in a cell (which, in this case, is called *atom* in LISP lingo). For example, you divide 1 by 3, and the interpreter, by understanding that $\frac{1}{3}$ is an irreducible fraction[11], creates a cell with 1 and 3 numbers. Some time after, you may multiply this cell by 6, and the multiplication function inside LISP interpreter may return much better result (2 without *noise*).

Printing function in interpreter can also print something like `1 / 3` instead of floating point number.

This is sometimes called “fractional arithmetic” [see TAOCP[12], 3rd ed., (1997), 4.5.1, page 330].

This is not symbolic computation in any way, but this is slightly better than storing ratios/fractions as just floating point numbers.

Drawbacks are clearly visible: you need more memory to store ratio instead of a number; and all arithmetic functions are more complex and slower, because they must handle both numbers and ratios.

Perhaps, because of drawbacks, some programming languages offers separate (*rational*) data type, as language feature, or supported by a library [13]: Haskell, OCaml, Perl, Ruby, Python (*fractions*), Smalltalk, Java, Clojure, C/C++[14].

[4] *Symbol* like in LISP
[5] https://en.wikipedia.org/wiki/Symbolic_integration
[6] https://en.wikipedia.org/wiki/Loss_of_significance
[7] https://en.wikipedia.org/wiki/Round-off_error
[8] Floating-point unit
[9] Central processing unit
[10] https://en.wikipedia.org/wiki/Rational_data_type
[11] https://en.wikipedia.org/wiki/Irreducible_fraction
[12] The Art Of Computer Programming (Donald Knuth's book)
[13] More detailed list: https://en.wikipedia.org/wiki/Rational_data_type
[14] By GNU Multiple Precision Arithmetic Library

Chapter 13

Graph theory

Graph is a group of nodes, some of them may be connected with each other, some are not. One of the popular examples is the map of country: there are cities and roads. ”Cities” are called ”nodes” or ”vertices” in mathematics lingo, while ”roads” are called ”edges”. Another popular example of graph is computer network, including Internet. Computer network is graph indeed, but it’s closer to ”sparse graph”, because for the most part, computer networks are trees.

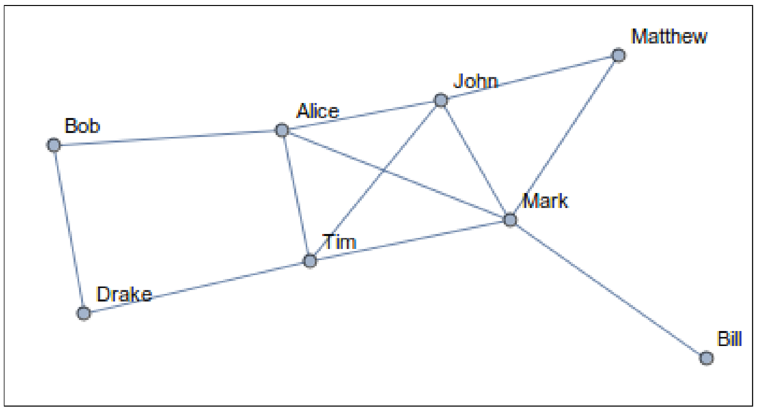
13.1 Clique in graph theory

”Clique” in everyday speech (especially in political news) denotes a tight-knit group of people inside of some community. In graph theory, ”clique” is a subgraph (part of graph) each vertices (”nodes” or ”members”) of which are connected with each other.

13.1.1 Social graph: simple example

”Social graph” is a graph representing social links. Here is example I made in Wolfram Mathematica:

```
community =
 Graph[{John <-> Mark, John <-> Alice, Mark <-> Alice, Tim <-> Alice,
   Matthew <-> John, Matthew <-> Mark, Tim <-> John, Drake <-> Tim,
   Bob <-> Drake, Bill <-> Mark, Bob <-> Alice, Tim <-> Mark},
  VertexLabels -> "Name"]
```

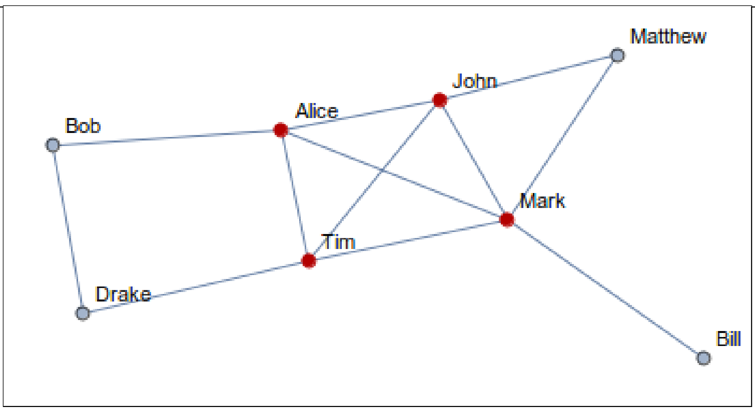


Let’s try to find largest clique:

```
In[]:= clique = FindClique[community]
Out[]= {{John, Mark, Alice, Tim}}
```

Indeed, each of these four persons is connected to each among other 3. Wolfram Mathematica can highlight subgraph in graph:

```
HighlightGraph[community, clique]
```



13.1.2 Social graph: IRC network

Internet Relay Chat (IRC) is popular among open-source developers. One of the most popular IRC networks is Freenode. And one of the most crowded IRC channel there is #ubuntu, devoted to Ubuntu Linux. I used data from it, because all logs are available (starting at 2004), for example: http://irclogs.ubuntu.com/2015/01/01/%23ubuntu.txt.

When someone asks, and someone another going to answer the question, IRC users are address each other in this way:

```
[00:11] <synire> How would one find the path of an application installed using
    terminal?
[00:11] <zykotick9> synire: "whereis foo"
[00:11] <synire> zykotick9: thanks!
```

It's not a rule, but well-established practice, so we can recover the information, which users talks to which users most often. Let's say, we would build a link between two IRC users if 1) they talk to each other at least 10-11 days (not necessary consequent); 2) do this at least 6 months (not necessary consequent).

The largest cliques of #ubuntu IRC channel in 10-11 years period are these:

```
* clique size 11
['ubottu', 'ActionParsnip', 'ikonia', 'Ben64', 'zykotick9', 'theadmin', 'dr_willis',
    'MonkeyDust', 'usr13', 'bekks', 'iceroot']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'bazhang', 'Pici', 'iceroot', 'theadmin
    ', 'IdleOne', 'erUSUL']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'bazhang', 'Pici', 'iceroot', 'theadmin
    ', 'zykotick9', 'usr13']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'bazhang', 'Pici', 'iceroot', '
    sebsebseb', 'IdleOne', 'erUSUL']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'Dr_Willis', 'Pici', 'edbian', 'IdleOne
    ', 'Jordan_U', 'theadmin']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'Dr_Willis', 'Pici', 'edbian', 'IdleOne
    ', 'Jordan_U', 'sebsebseb']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'Dr_Willis', 'Pici', 'erUSUL', 'iceroot
    ', 'IdleOne', 'theadmin']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'Dr_Willis', 'Pici', 'erUSUL', 'iceroot
    ', 'IdleOne', 'sebsebseb']
* clique size 10
['ubottu', 'ActionParsnip', 'ikonia', 'jrib', 'Dr_Willis', 'Pici', 'erUSUL', 'iceroot
    ', 'ubuntu', 'sebsebseb']
* clique size 10
```

```
['ubottu', 'ActionParsnip', 'ikonia', 'Ben64', 'histo', 'bekks', 'MonkeyDust', '
    dr_willis', 'iceroot', 'usr13']
...
```

Perhaps, these users are frequenters of the channel. List of all cliques are here: https://αβγ.ελ/current_tree/graph/clique/files/IR
The output is not terse, because all listed cliques are cliques indeed, and single user or users group can be member of several cliques, that's correct. Cliques can be overlapped and be members of bigger cliques. It's possible to produce more human-like results using more complex algorithms for finding communities.

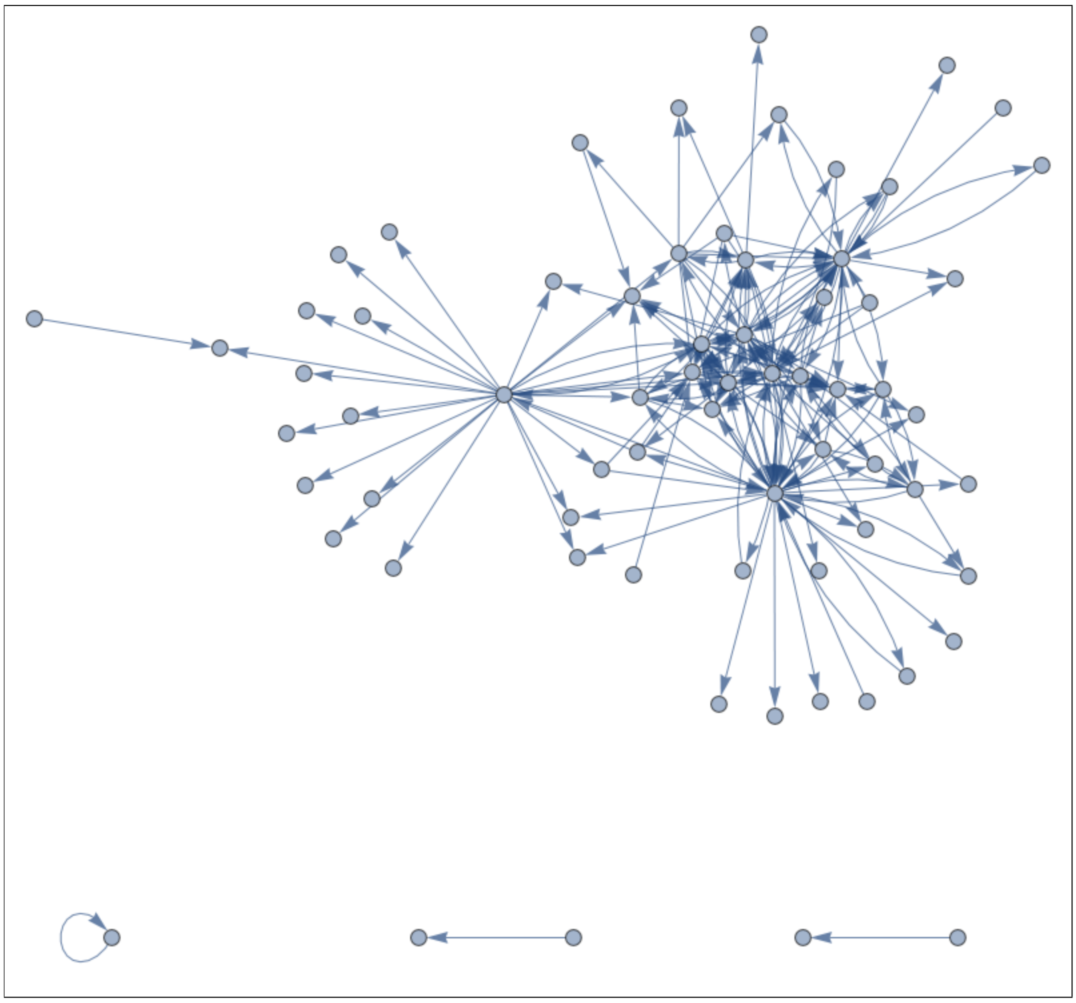
The source code of my scripts here: https://αβγ.ελ/current_tree/graph/clique/files/IRC. I used the excellent networkx graph library.

13.1.3 Attempt to find communities in IRC social graph

Wolfram Mathematica can try to find communities within social graph. Here I will import information about all IRC interactions from the start of 2013 till the summer of 2015. User nicknames are coded by numbers for simplicity.

```
In[]:= g2 =
 Graph[{91708 -> 93574, 93414 -> 91525, 93414 -> 89579,
   90407 -> 93896, 93414 -> 93598, 93809 -> 5909, 93698 -> 93801,
   93163 -> 83317, 84930 -> 93896, 93414 -> 92947, 93414 -> 91708,
   93792 -> 92887, 84930 -> 91708, 91708 -> 84930, 88400 -> 93698,
   ...
   93809 -> 93475, 93698 -> 92887, 93801 -> 93670, 92887 -> 93598}]
```

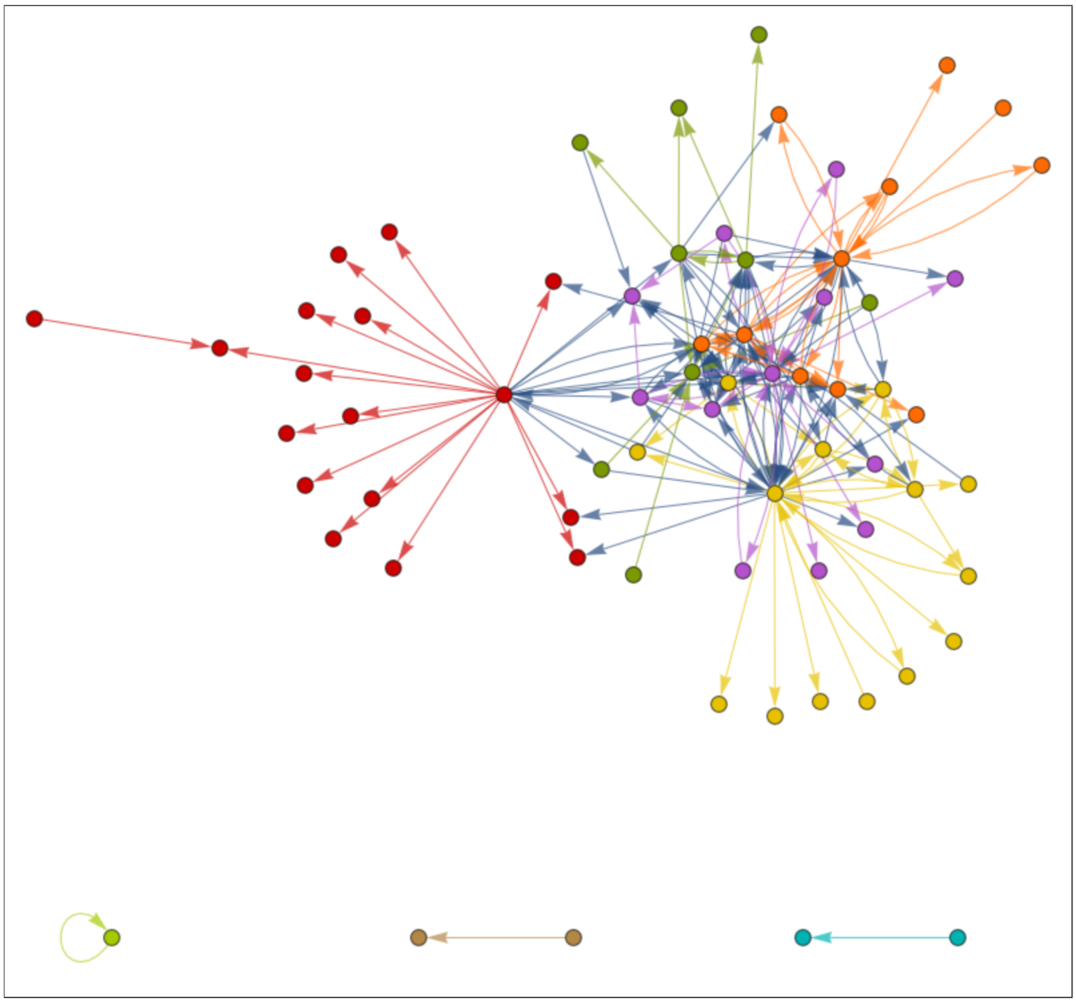
The resulting graph is:



There some artifacts (at the bottom) which can be ignored so far, I think. There is prominent centers: one huge and two others are smaller. I'm not sure, but I can suggest these parts of graph are just users who has different sleep patterns, or, more likely, from different time zones, so each important time zone (like Americas, Europe, Asia/Oceania) may have their own social communities. But again, I'm not sure, this should be investigated first.

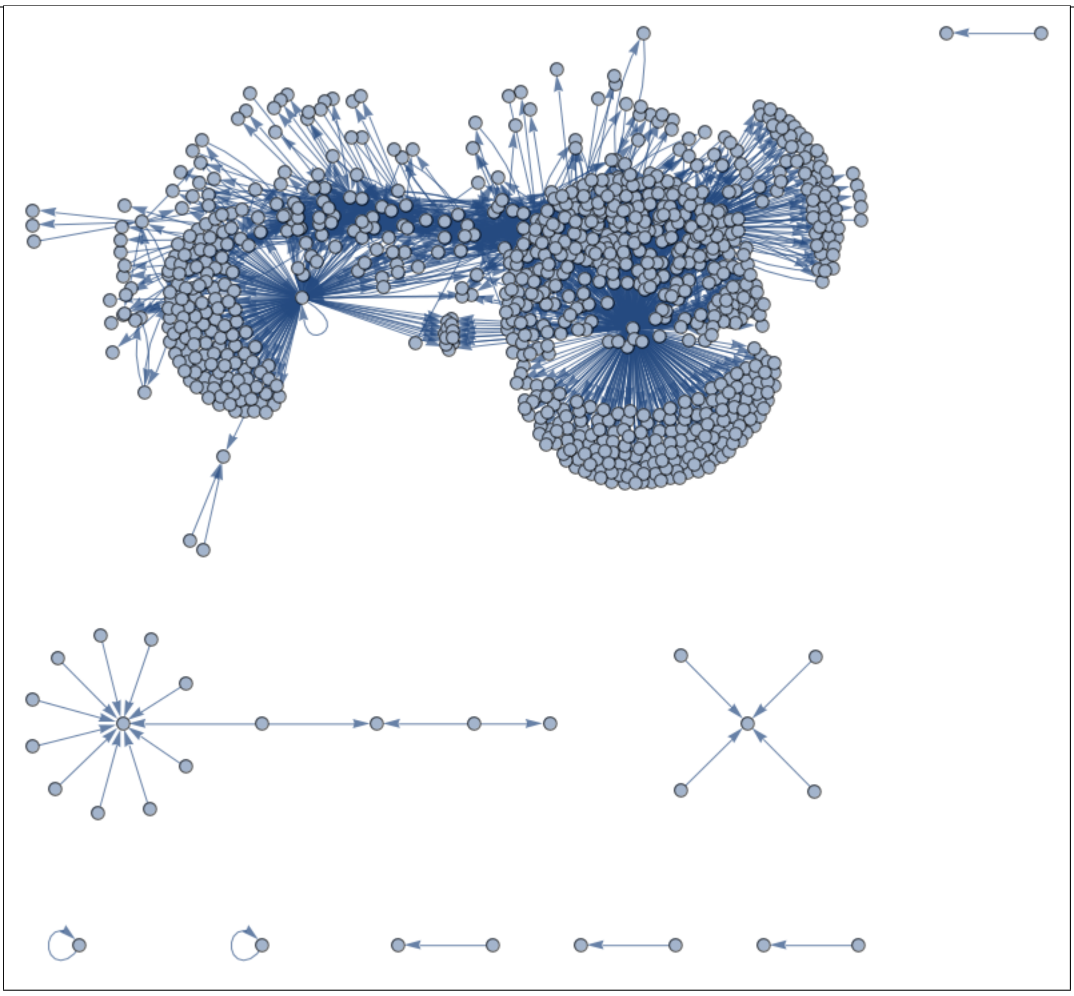
Let's try to find communities and hightlight them within the graph:

```
c2 = FindGraphCommunities[g2];
HighlightGraph[g2, Map[Subgraph[g2, #] &, c2]]
```



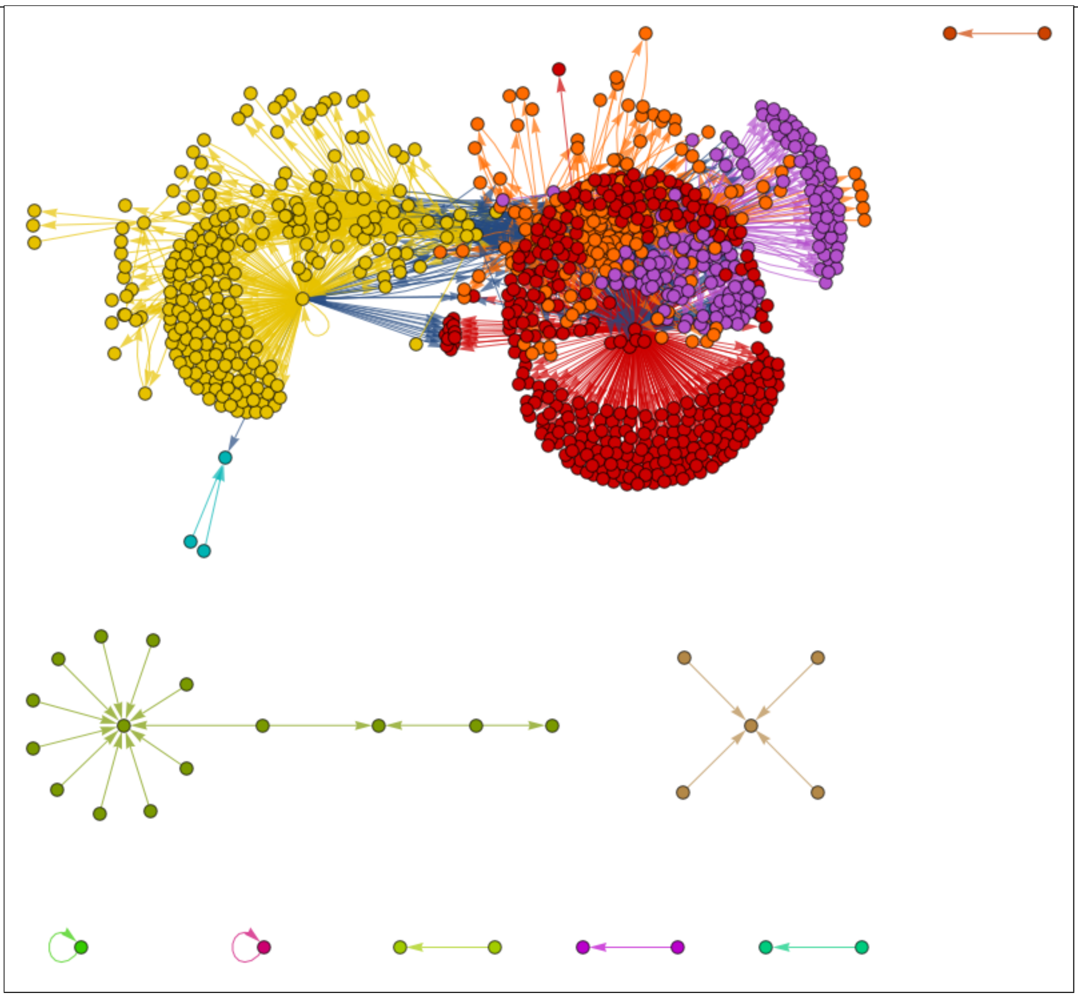
Hard to say if Mathematica right, but this is what it did.

Now let's take the whole graph of all IRC interactions starting at year 2004 till the summer of 2015. The graph is much bigger:



There are more artifacts.

Let's apply Mathematica's method to find communities:



Is it right? Maybe. Needless to say, since timespan is so long (at least 10 years), we can believe that some communities which may exists in 2004-2006 may be extinct in 2014-2015 (people got older, lost their interest in Ubuntu Linux, etc), but they all are visible on this graph.

Summary: perhaps, on our next experiment we should filter out IRC data by years and time zones.

13.1.4 Social graph: social networks

Perhaps, social networking websites like Facebook and Twitter in the "people you may know" tab shows you users of most populous (by your current friends) cliques. It may be much more complex in reality, but nevertheless, this is simplest possible way to offer you new social contacts.

13.1.5 Links graph: Wikipedia

Wikipedia has a lot of internal links, 463,000,000 in English Wikipedia as of summer 2015, if not to count user/talk/-media pages, etc. It's possible to build a graph where Wikipedia article is a vertice (or node) and a link from one article to another is edge. By link between articles we would call the case when the first article has the link to the second article, but also the second has the link to the first one.

Here are some examples of cliques I found this way. Number in parenthesis is clique size.

- Chess-related articles (9): Reuben Fine, Mikhail Botvinnik, Samuel Reshevsky, Max Euwe, FIDE, Alexander Alekhine, World Chess Championship, José Raúl Capablanca, AVRO 1938 chess tournament.
- Utah-related articles (9): Red Line (TRAX), Utah Transit Authority, Blue Line (TRAX), TRAX (light rail), Salt Lake City, Green Line (TRAX), FrontRunner, University of Utah, Utah.
- Articles related to Doctor Who (9): Doctor Who (film), Doctor Who, The Doctor (Doctor Who), Eighth Doctor, The Master (Doctor Who), Gallifrey, TARDIS, Doctor Who Magazine, Seventh Doctor.
- Space (9): New Horizons, Pioneer 11, Voyager 1, Europa (moon), Callisto (moon), Ganymede (moon), Jupiter, Io (moon), Pioneer 10.

- Hip hop music (9): G-funk, Dr. Dre, Death Row Records, Snoop Dogg, The Chronic, Gangsta rap, West Coast hip hop, N.W.A, Hip hop music.
- Metal music (9): Master of Puppets, Thrash metal, Cliff Burton, James Hetfield, Kirk Hammett, Metallica, Kill 'Em All, Ride the Lightning, Dave Mustaine.
- The Beatles (8): Break-up of the Beatles, The Beatles, George Harrison, Let It Be, John Lennon, Paul McCartney, Ringo Starr, Abbey Road.

Each Wikipedia article within any of these cliques has links to each article in clique.

Full lists of first 1000 largest cliques in English, Russian and Ukrainian Wikipedias plus source code of my scripts is here: https://αβγ.ελ/current_tree/graph/clique/files/wikipedia.

13.1.6 Social graph: LiveJournal spammers

LiveJournal was a popular blogging platform in Russian-speaking Internet, which, as any other platform, flooded by spammers. I once tried, for experiment, to find a way to make distinction between them and human users. (I did this in 2010-2011, so this information may be not relevant these days.)

Aside of false texts spammers posted to their blogs, spammers also mutually friended may spam accounts, so it was not unusual to register, let's say, 1000 fake accounts and friend each other.

If to build a graph of all links between LiveJournal users, and find largest cliques, there will be prominent unusually large cliques of LiveJournal users, up to 1000. In real world, you would not easily find a social group of 1000 persons who keeps mutual links with each other (there is interesting reading about it: Dunbar's number).

Well, spammers could lower this number, so each fake user would have 100-200 mutual friends instead of 1000 (which is less suspicious), but still, cliques were too perfect: each node connected to each other with very low amount of "external" links, leading to other spammer's cliques and human users.

13.1.7 Links graph: link farms

Talking about spammers, there was (or maybe still used today?) also a Black Hat SEO method to build "link farms": this is a collection of many websites which has links to each other. Interestingly, if you analyze link graph and find cliques, such farms are clearly visible.

13.1.8 Further reading

"SAT/SMT by Example"[1] has a short example, on how to find max cliques using MaxSAT.

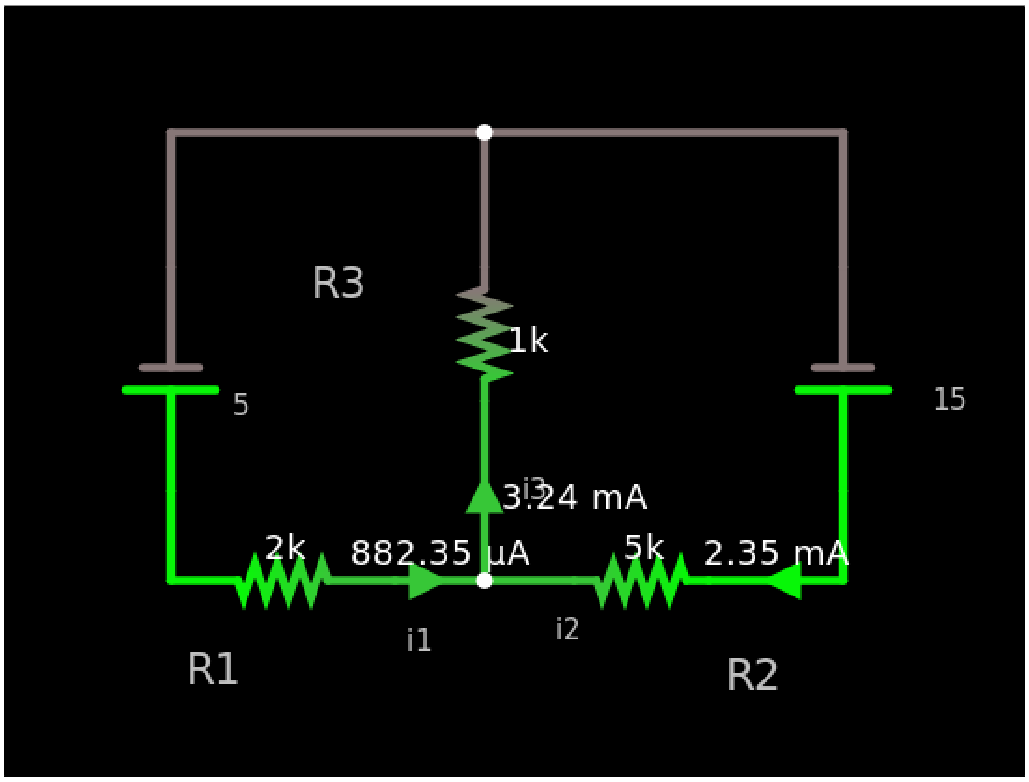
[1] https://sat-smt.codes/

Chapter 14

Linear algebra

14.1 Gaussian elimination: Kirchhoff's circuit laws

The circuit I've created on falstad.com[1]:



Click here to open it on their website and run: http://tinyurl.com/y8raoud3.

The problem: find all 3 current values in 2 loops. This is usually solved by solving a system of linear equations.

Overkill, but Z3 SMT-solver can be used here as well, since it can solve linear equations as well, over real numbers:

```
from z3 import *

i1, i2, i3 = Reals ("i1 i2 i3")

R1=2000
R2=5000
R3=1000

V1=5 # left
```

[1]http://falstad.com/circuit/

```
V2=15 # right

s=Solver()

s.add(i3 == i1+i2)

s.add(V1 == R1*i1 + R3*i3)
s.add(V2 == R2*i2 + R3*i3)

print s.check()
m=s.model()
print m
print m[i1].as_decimal(6)
print m[i2].as_decimal(6)
print m[i3].as_decimal(6)
```

And the result:

```
sat
[i3 = 11/3400, i1 = 3/3400, i2 = 1/425]
0.000882?
0.002352?
0.003235?
```

Same as on falstad.com online simulator.

Z3 represents real numbers as fractions, then we convert them to numerical form...

Further work: take a circuit as a graph and build a system of equations.

14.1.1 Gaussian elimination

SMT-solver is overkill, these linear equations can be solved using simple and well-known Gaussian elimination.

First, we rewrite the system of equation:

```
   i1 +     i2 -     i3 == 0
R1*i1 +           R3*i3 == V1
        R2*i2 + R3*i3 == V2
```

Or in matrix form:

```
[ 1,    1,    -1    | 0  ]
[ 2000, 0,    1000  | 5  ]
[ 0,    5000, 1000  | 15 ]
```

I can solve it using Wolfram Mathematica, using RowReduce [2].

```
In[1]:= RowReduce[{{1, 1, -1, 0}, {2000, 0, 1000, 5}, {0, 5000, 1000, 15}}]
Out[1]= {{1,0,0,3/3400},{0,1,0,1/425},{0,0,1,11/3400}}

In[2]:= 3/3400//N
Out[2]= 0.000882353

In[3]:= 1/425//N
Out[3]= 0.00235294

In[4]:= 11/3400//N
Out[4]= 0.00323529
```

This is the same result: i1, i2 and i3 in numerical form.

ReduceRow's output is:

[2] http://reference.wolfram.com/language/ref/RowReduce.html

```
[ 1,0,0 | 3/3400  ]
[ 0,1,0 | 1/425   ]
[ 0,0,1 | 11/3400 ]
```

... back to expressions, this is:

```
1*i1 + 0*i2 + 0*i3 = 3/3400
0*i1 + 1*i2 + 0*i3 = 1/425
0*i1 + 0*i2 + 1*i3 = 11/3400
```

In other words, this is just what i1/i2/i3 are.

Now something down-to-earth, C example I've copypasted from Rosetta Code [3], working with no additional libraries, etc:

```
// copypasted from https://rosettacode.org/wiki/Gaussian_elimination#C

#include <stdio.h>
#include <stdlib.h>
#include <math.h>

#define mat_elem(a, y, x, n) (a + ((y) * (n) + (x)))

void swap_row(double *a, double *b, int r1, int r2, int n)
{
        double tmp, *p1, *p2;
        int i;

        if (r1 == r2) return;
        for (i = 0; i < n; i++) {
                p1 = mat_elem(a, r1, i, n);
                p2 = mat_elem(a, r2, i, n);
                tmp = *p1, *p1 = *p2, *p2 = tmp;
        }
        tmp = b[r1], b[r1] = b[r2], b[r2] = tmp;
}

void gauss_eliminate(double *a, double *b, double *x, int n)
{
#define A(y, x) (*mat_elem(a, y, x, n))
        int i, j, col, row, max_row,dia;
        double max, tmp;

        for (dia = 0; dia < n; dia++) {
                max_row = dia, max = A(dia, dia);

                for (row = dia + 1; row < n; row++)
                        if ((tmp = fabs(A(row, dia))) > max)
                                max_row = row, max = tmp;

                swap_row(a, b, dia, max_row, n);

                for (row = dia + 1; row < n; row++) {
                        tmp = A(row, dia) / A(dia, dia);
                        for (col = dia+1; col < n; col++)
                                A(row, col) -= tmp * A(dia, col);
                        A(row, dia) = 0;
                        b[row] -= tmp * b[dia];
                }
        }
```

[3] https://rosettacode.org/wiki/Gaussian_elimination#C

```
        for (row = n - 1; row >= 0; row--) {
                tmp = b[row];
                for (j = n - 1; j > row; j--)
                        tmp -= x[j] * A(row, j);
                x[row] = tmp / A(row, row);
        }
#undef A
}

int main(void)
{
        double a[] = {
                1, 1, -1,
                2000, 0, 1000,
                0, 5000, 1000
        };
        double b[] = { 0, 5, 15 };
        double x[3];
        int i;

        gauss_eliminate(a, b, x, 3);

        for (i = 0; i < 3; i++)
                printf("%g\n", x[i]);

        return 0;
}
```

I run it:

```
0.000882353
0.00235294
0.00323529
```

See also: https://en.wikipedia.org/wiki/Gaussian_elimination,
http://mathworld.wolfram.com/GaussianElimination.html.

But a fun with SMT solver is that we can solve these equations without any knowledge of linear algebra, matrices, Gaussian elimination and whatnot.

According to the source code of Z3, it can perform Gaussian Elimination, perhaps, whenever it can do so.

Some people try to use Z3 to solve problems with operational amplifiers: 1, 2.

Chapter 15

Acronyms used

GCD	Greatest Common Divisor	2
LCM	Least Common Multiple	5
LCG	Linear congruential generator	61
CPU	Central processing unit	136
FPU	Floating-point unit	136
PRNG	Pseudorandom number generator	57
CRC	Cyclic redundancy check	99
NSA	National Security Agency	23
RSA	Rivest–Shamir–Adleman cryptosystem	47
TAOCP	The Art Of Computer Programming (Donald Knuth's book)	136
MSB	Most Significant Bit	6
MAC	Message authentication code	93
SIMD	Single Instruction, Multiple Data	17
RAM	Random-Access Memory	17
RAID	Redundant Array of Independent Disks	16
ASCII	American Standard Code for Information Interchange	19
AKA	Also Known As	23
NSA	National Security Agency	23

MSB	Most Significant Bit	6
ROM	Read-Only Memory	20